On elliptic systems involving critical Hardy-Sobolev exponents*

X. Zhong & W. Zou

Department of Mathematical Sciences, Tsinghua University,

Beijing 100084, China

Abstract

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be an open domain which is not necessarily bounded. By using variational methods, we consider the following elliptic systems involving multiple Hardy-Sobolev critical exponents:

$$\begin{cases} -\Delta u - \lambda \frac{|u|^{2^*(s_1)-2}u}{|x|^{s_1}} = \kappa\alpha \frac{1}{|x|^{s_2}}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ -\Delta v - \mu \frac{|v|^{2^*(s_1)-2}v}{|x|^{s_1}} = \kappa\beta \frac{1}{|x|^{s_2}}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ (u,v) \in \mathscr{D} := D_0^{1,2}(\Omega) \times D_0^{1,2}(\Omega), \end{cases}$$

where $s_1, s_2 \in (0,2), \alpha > 1, \beta > 1, \lambda > 0, \mu > 0, \kappa \neq 0, \alpha + \beta \leq 2^*(s_2)$. Here, $2^*(s) := \frac{2(N-s)}{N-2}$ is the critical Hardy-Sobolev exponent. We mainly study the critical case (i.e., $\alpha + \beta = 2^*(s_2)$) when Ω is a cone (in particular, $\Omega = \mathbb{R}^N_+$ or $\Omega = \mathbb{R}^N$). We will establish a sequence of fundamental results including regularity, symmetry, existence and multiplicity, uniqueness and nonexistence, *etc.* In particular, the sharp constant and extremal functions to the following kind of double-variable inequalities

$$S_{\alpha,\beta,\lambda,\mu}(\Omega)\Big(\int_\Omega \big(\lambda\frac{|u|^{2^*(s)}}{|x|^s} + \mu\frac{|v|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa\frac{|u|^\alpha|v|^\beta}{|x|^s}\big)dx\Big)^{\frac{2}{2^*(s)}} \leq \int_\Omega \big(|\nabla u|^2 + |\nabla v|^2\big)dx$$

for $(u,v) \in \mathscr{D}$ will be explored. Further results about the sharp constant $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ with its extremal functions when Ω is a general open domain will be involved.

Key words: Elliptic system, sharp constant, Hardy-Sobolev exponent, existence, nonexistence, ground state solution, infinitely many sign-changing solutions.
Mathematics Subject Classification (2010): 35B38; 35J10; 35J15; 35J20; 35J60; 49J40.

*Supported by NSFC(11371212, 11271386). E-mail: zhongxuexiu1989@163.com wzou@math.tsinghua.edu.cn

Content

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) be an open domain which is not necessarily bounded. We study the following nonlinear elliptic systems

$$\begin{cases} -\Delta u - \lambda \frac{|u|^{2^*(s_1)-2}u}{|x|^{s_1}} = \kappa\alpha \frac{1}{|x|^{s_2}}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ -\Delta v - \mu \frac{|v|^{2^*(s_1)-2}v}{|x|^{s_1}} = \kappa\beta \frac{1}{|x|^{s_2}}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ (u,v) \in \mathscr{D} := D_0^{1,2}(\Omega) \times D_0^{1,2}(\Omega), \end{cases} \tag{1.1}$$

where $s_1, s_2 \in (0,2), \alpha > 1, \beta > 1, \lambda > 0, \mu > 0, \kappa \neq 0, \alpha+\beta \leq 2^*(s_2) := \frac{2(N-s_2)}{N-2}$.

The interest in studying the nonlinear Schrödinger systems is motivated by real problems in nonlinear optics, plasma physics, condensed matter physics, etc. For example, the coupled nonlinear Schrödinger systems arise in the description of several physical phenomena such as the propagation of pulses in birefringent optical fibers and Kerr-like photorefractive media, see [2, 11, 16, 23, 24, 28], etc. The problem comes from the physical phenomenon with a clear practical significance. The researches on solutions under different situations not only corresponds to different physical interpretation, but also has a pure mathematical theoretical significance. Hence, the coupled nonlinear Schrödinger systems are widely studied in recently years, we refer the readers to [1, 3, 20, 22, 25] and the references therein.

For any $s \in [0,2]$, we define the measure $d\mu_s := \frac{1}{|x|^s}dx$ and $\|u\|_{p,s}^p := \int_\Omega |u|^p d\mu_s$. We also use the notation $\|u\|_p := \|u\|_{p,0}$. The Hardy-Sobolev inequality[5, 7, 14] asserts that $D_0^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*(s)}(\mathbb{R}^N, d\mu_s)$ is a continuous embedding for $s \in [0,2]$. For a general open domain Ω, there exists a positive constant $C(s,\Omega)$ such that

$$\int_\Omega |\nabla u|^2 dx \geq C(s,\Omega)\Big(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx\Big)^{\frac{2}{2^*(s)}}, \quad u \in D_0^{1,2}(\Omega).$$

Define $\mu_{s_1}(\Omega)$ as

$$\mu_{s_1}(\Omega) := \inf\Big\{\frac{\int_\Omega |\nabla u|^2 dx}{(\int_\Omega \frac{|u|^{2^*(s_1)}}{|x|^{s_1}}dx)^{\frac{2}{2^*(s_1)}}} : \quad u \in D_0^{1,2}(\Omega)\backslash\{0\}\Big\}. \tag{1.2}$$

Consider the case of $\Omega = \mathbb{R}_+^N$, it is well known that the extremal function of $\mu_{s_1}(\mathbb{R}_+^N)$ is parallel to the ground state solution of the following problem:

$$\begin{cases} -\Delta u = \frac{|u|^{2^*(s_1)-2}u}{|x|^{s_1}} & \text{in } \mathbb{R}_+^N, \\ u = 0 \text{ on } \partial\mathbb{R}_+^N. \end{cases} \tag{1.3}$$

We note that the existence of ground state solution of (1.3) for $0 < s_1 < 2$ is solved by Ghoussoub and Robert [13]. They also gave some properties about the regularity, symmetry and decay estimates. The instanton $U(x) := C\big(\kappa + |x|^{2-s_2}\big)^{-\frac{N-2}{2-s_2}}$ for

$0 \leq s_2 < 2$ is a ground state solution to (1.4) below (see [18] and [27]):

$$\begin{cases} \Delta u + \frac{u^{2^*(s_2)-1}}{|x|^{s_2}} = 0 & \text{in } \mathbb{R}^N, \\ u > 0 \ \text{ in } \mathbb{R}^N \quad \text{and} & u \to 0 \text{ as } |x| \to +\infty. \end{cases} \tag{1.4}$$

The case that $0 \in \partial\Omega$ has become an interesting topic in recent years since the curvature of $\partial\Omega$ at 0 plays an important role, see [9, 12, 13, 15], etc. A lot of sufficient conditions are given in order to ensure that $\mu_{s_1}(\Omega) < \mu_{s_1}(\mathbb{R}^N_+)$ in those papers. It is standard to apply the blow-up analysis to show that $\mu_{s_1}(\Omega)$ can be achieved by some positive $u \in H^1_0(\Omega)$ (e.g., see [12, Corollary 3.2]), which is a ground state solution of

$$\begin{cases} -\Delta u = \frac{|u|^{2^*(s_1)-2}u}{|x|^{s_1}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and the least energy equals $\big(\dfrac{1}{2} - \dfrac{1}{2^*(s_1)}\big)\mu_{s_1}(\Omega)^{\frac{N-s_1}{2-s_1}}$.

However, it seems there is no article before involving the system case like (1.1) with Hardy-Sobolev critical exponents, which we are going to deal with in the current paper. It is well known that the main difficulty is the lack of compactness inherent in these problems involving Hardy-Sobolev critical exponents. The compactness concentration argument (see [21], etc.) is a powerful tool to handle with these critical problems. It is also well known that the compactness concentration argument depends heavily on the limit problem. Consider a bounded domain Ω, if $0 \notin \bar{\Omega}$, we see that $\frac{1}{|x|^{s_i}}, i = 1, 2$ are regular. We are interested in the case of that $0 \in \bar{\Omega}$. It is easy to see that when $0 \in \Omega$, the limit domain is $\mathbb{R}^N$, and when $0 \in \partial\Omega$, the limit domain is usually a cone. Especially, when $\partial\Omega$ possesses a suitable regularity (e.g. $\partial\Omega \in C^2$ at $x = 0$), the limit domain is $\mathbb{R}^N_+$ after a suitable rotation. Hence, in present paper, we mainly study the critical elliptic systems (1.1) with $\alpha + \beta = 2^*(s_2)$ and Ω is a cone.

Definition 1.1. *A cone in $\mathbb{R}^N$ is an open domain Ω with Lipschitz boundary and such that $tx \in \Omega$ for every $t > 0$ and $x \in \Omega$.*

We will establish a sequence of fundamental results to the system (1.1) including regularity, symmetry, existence and multiplicity, and nonexistence, *etc.* Since there are a large number of conclusions in the current paper, we do not intend to list them here. This paper is organized as follows:

In Section 2, we will establish by a direct method a type of interpolation inequalities, which are essentially the variant Caffarelli-Kohn-Nirenberg (CKN) inequalities, see [5].

In Section 3, we will study the regularity, symmetry and decay estimation about the nonnegative solutions of (1.1). Taking $\mathbb{R}^N_+$ as a specific example, we will study the regularity based on the technique of Moser's iteration (see Proposition 3.1). By the method of moving planes, we obtain the symmetry result (see Proposition 3.3). Due to the Kelvin transformation, we get the decay estimation (see Proposition 3.2).

In Section 4, we shall study the basic properties of the corresponding Nehari manifold.

In Section 5, we will give a nonexistence of nontrivial ground state solution of (1.1) for the case $s_2 \geq s_1$, see Theorem 5.1.

In Section 6, we will give an existence result of positive solution result for a special case : $\lambda = \mu(\frac{\beta}{\alpha})^{\frac{2^*(s_1)-2}{2}}$, see Corollary 6.1. Further, we prepare a sequence of preliminaries for the existence result which are not only useful for us to study the case of $s_1 = s_2$ in Section 7, but also the case of $s_1 \neq s_2$ in Section 8.

In Section 7, we will focus on the case of $s_1 = s_2 = s \in (0,2)$ when Ω is a cone. In this case, the nonlinearities are homogeneous which enable us to define the following constant

$$S_{\alpha,\beta,\lambda,\mu}(\Omega) := \inf_{(u,v)\in\widetilde{\mathscr{D}}} \frac{\int_\Omega \big(|\nabla u|^2 + |\nabla v|^2\big)dx}{\Big(\int_\Omega \big(\lambda\frac{|u|^{2^*(s)}}{|x|^s} + \mu\frac{|v|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa\frac{|u|^\alpha|v|^\beta}{|x|^s}\big)dx\Big)^{\frac{2}{2^*(s)}}}, \tag{1.5}$$

where

$$\widetilde{\mathscr{D}} := \{(u,v)\in\mathscr{D}:\ \int_\Omega \big(\lambda\frac{|u|^{2^*(s)}}{|x|^s} + \mu\frac{|v|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa\frac{|u|^\alpha|v|^\beta}{|x|^s}\big)dx > 0\}. \tag{1.6}$$

In particular, we shall see that $\widetilde{\mathscr{D}} = \mathscr{D}\backslash\{(0,0)\}$ if and only if

$$\kappa > -(\frac{\lambda}{\alpha})^{\frac{\alpha}{2^*(s)}}(\frac{\mu}{\beta})^{\frac{\beta}{2^*(s)}},$$

see Lemma 7.2. When $\kappa < 0$, we will prove that $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ has no nontrivial extremals (see Lemma 7.2). Hence, we will mainly focus on the case of $\kappa > 0$ and show that the system (1.1) possesses a least energy solution and that $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ is achieved. These conclusions will produce the sharp constant and extremal functions to the following kind of inequalities with double-variable

$$S_{\alpha,\beta,\lambda,\mu}(\Omega)\Big(\int_\Omega \big(\lambda\frac{|u|^{2^*(s)}}{|x|^s} + \mu\frac{|v|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa\frac{|u|^\alpha|v|^\beta}{|x|^s}\big)dx\Big)^{\frac{2}{2^*(s)}} \\ \leq \int_\Omega \big(|\nabla u|^2 + |\nabla v|^2\big)dx$$

for $(u,v)\in\mathscr{D}$. For this purpose, a kind of Pohozaev identity will be established. Then the existence, regularity, uniqueness and nonexistence results of the positive ground state solution to the system (1.1) can be seen in this section. Under some proper hypotheses, we will show that the positive ground state solution must be of the form $\big(C(t_0)U, t_0C(t_0)U\big)$, where $t_0 > 0$ and $C(t_0)$ can be formulated explicitly and U is the ground state solution of

$$\begin{cases} -\Delta u = \mu_s(\Omega)\frac{u^{2^*(s)-1}}{|x|^s} & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Taking a special case $N=3, s=1, \alpha=\beta=2, \lambda=\mu=2\kappa$ in consideration, we will find out all the positive ground state solutions to (1.1) . Based on these conclusions, we may prove the existence of infinitely many sign-changing solutions of the system (1.1) on a cone Ω by gluing together suitable signed solutions corresponding to each sub-cone. Further, if Ω is a general open domain, the sharp constant $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ and its extremal functions will be investigated. We will find a way to compute $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ and to judge when $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ can be achieved if Ω is a general open domain.

In Section 8, the system (1.1) satisfying $s_1 \neq s_2 \in (0,2)$ will be studied. We shall consider a new approximation to the original system (1.1). The estimation on the least energy and the positive ground state along with its geometric structure to the approximation will be established. Finally, the existence of positive ground state solution to the original system will be given.

2 Interpolation inequalities

For $s_1 \neq s_2$, we note that there is no embedding relationship between $L^{2^*(s_1)}(\Omega, d\mu_{s_1})$ and $L^{2^*(s_2)}(\Omega, d\mu_{s_2})$ for any domain Ω with $0 \in \bar{\Omega}$. Hence, we are going to establish some interpolation inequalities in this section.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be an open set. Assume that $0 \leq s_1 < s_2 < s_3 \leq 2$, then there exists $\theta = \frac{(N-s_1)(s_3-s_2)}{(N-s_2)(s_3-s_1)} \in (0,1)$ such that*

$$|u|_{2^*(s_2),s_2} \leq |u|^{\theta}_{2^*(s_1),s_1} |u|^{1-\theta}_{2^*(s_3),s_3} \tag{2.1}$$

for all $u \in L^{2^*(s_1)}(\Omega, \frac{dx}{|x|^{s_1}}) \cap L^{2^*(s_3)}(\Omega, \frac{dx}{|x|^{s_3}})$.

Proof. Define $\varrho = \frac{s_3-s_2}{s_3-s_1}$, then $1-\varrho = \frac{s_2-s_1}{s_3-s_1}$. A direct calculation shows that

$$s_2 = \varrho s_1 + (1-\varrho) s_3 \tag{2.2}$$

and

$$2^*(s_2) = \varrho 2^*(s_1) + (1-\varrho) 2^*(s_3). \tag{2.3}$$

It follows from the Hölder inequality that

$$\begin{aligned}\int_\Omega \frac{|u|^{2^*(s_2)}}{|x|^{s_2}} dx =& \int_\Omega \Big(\frac{|u|^{2^*(s_1)}}{|x|^{s_1}}\Big)^{\varrho} \Big(\frac{|u|^{2^*(s_3)}}{|x|^{s_3}}\Big)^{1-\varrho} dx \\ \leq& \Big(\int_\Omega \frac{|u|^{2^*(s_1)}}{|x|^{s_1}} dx\Big)^{\varrho} \Big(\int_\Omega \frac{|u|^{2^*(s_3)}}{|x|^{s_3}} dx\Big)^{1-\varrho}.\end{aligned}$$

Let $\theta := \frac{2^*(s_1)}{2^*(s_2)} \varrho$, then by (2.3) again, $1-\theta = \frac{2^*(s_3)}{2^*(s_2)}(1-\varrho)$. Then we obtain that

$$|u|_{2^*(s_2),s_2} \leq |u|^{\theta}_{2^*(s_1),s_1} |u|^{1-\theta}_{2^*(s_3),s_3}$$

for all $u \in L^{2^*(s_1)}(\Omega, \frac{dx}{|x|^{s_1}}) \cap L^{2^*(s_3)}(\Omega, \frac{dx}{|x|^{s_3}})$, where

$$\theta = \frac{2^*(s_1)}{2^*(s_2)}\varrho = \frac{(N-s_1)(s_3-s_2)}{(N-s_2)(s_3-s_1)} \in (0,1)$$

has the following properties. Firstly, we note that $\theta > 0$ since $s_1 < s_2 < s_3 \leq 2 < N$. Secondly,

$$\theta < 1 \Leftrightarrow (N-s_1)(s_3-s_2) < (N-s_2)(s_3-s_1) \Leftrightarrow (s_2-s_1)(N-s_3) > 0.$$

□

Define

$$\vartheta(s_1, s_2) := \frac{N(s_2-s_1)}{s_2(N-s_1)} \quad \text{for } 0 \leq s_1 \leq s_2 \leq 2. \tag{2.4}$$

Corollary 2.1. *Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be an open set. Assume $0 \leq s_1 < 2$. Then for any $s_2 \in [s_1, 2]$ and $\theta \in [\vartheta(s_1, s_2), 1]$, there exists $C(\theta) > 0$ such that*

$$|u|_{2^*(s_1),s_1} \leq C(\theta)\|u\|^\theta |u|_{2^*(s_2),s_2}^{1-\theta} \tag{2.5}$$

for all $u \in D_0^{1,2}(\Omega)$, where $\|u\| := \big(\int_\Omega |\nabla u|^2 dx\big)^{\frac{1}{2}}$.

Proof. If $s_2 = s_1 = s$, then $\vartheta(s_1, s_2) = 0$ and (2.5) is a direct conclusion of Hardy-Sobolev inequality and the best constant $C(\theta) = \mu_s(\Omega)^{-\frac{\theta}{2}}, \forall\, \theta \in [0,1]$, where $\mu_s(\Omega)$ is defined by (1.2). If $s_1 = 0$, then $\vartheta(s_1, s_2) = 1, \theta = 1$ and (2.5) is just the well-known Sobolev inequality.

Next, we assume that $0 < s_1 < s_2 \leq 2$. We also note that if $\theta = 1$, (2.5) is just the well-known Sobolev inequality. Hence, next we always assume that $\theta < 1$. Define

$$\tilde{s} := s_2 - \frac{(N-s_2)(s_2-s_1)}{\theta(N-s_1) - (s_2-s_1)}.$$

Note $\theta \in [\vartheta(s_1, s_2), 1)$, we have that $0 \leq \tilde{s} < s_1 < s_2 \leq 2$. Then by Proposition 2.1, we have

$$|u|_{2^*(s_1),s_1} \leq |u|_{2^*(\tilde{s}),\tilde{s}}^\theta |u|_{2^*(s_2),s_2}^{1-\theta}.$$

Recalling the Hardy-Sobolev inequality, we have

$$|u|_{2^*(\tilde{s}),\tilde{s}} \leq \mu_{\tilde{s}}(\Omega)^{-\frac{1}{2}}\|u\|.$$

Hence, there exists a $C(\theta) > 0$ such that

$$|u|_{2^*(s_1),s_1} \leq C(\theta)\|u\|^\theta |u|_{2^*(s_2),s_2}^{1-\theta}.$$

□

Define

$$\varsigma(s_1, s_2) := \frac{(N-s_1)(2-s_2)}{(N-s_2)(2-s_1)} \quad \text{for } 0 \leq s_1 \leq s_2 \leq 2. \tag{2.6}$$

Corollary 2.2. *Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be an open set. Assume $0 < s_2 \leq 2$. Then for any $s_1 \in [0, s_2]$ and $\sigma \in [0, \varsigma(s_1, s_2)]$, there exists a $C(\sigma) > 0$ such that*

$$|u|_{2^*(s_2),s_2} \leq C(\sigma)\|u\|^{1-\sigma}|u|^{\sigma}_{2^*(s_1),s_1} \tag{2.7}$$

for all $u \in D_0^{1,2}(\Omega)$.

Proof. We only need to consider that case of $s_1 < s_2$ and $\sigma > 0$. Define

$$\bar{s} := s_1 + \frac{(N - s_1)(s_2 - s_1)}{(N - s_1) - (N - s_2)\sigma}.$$

Recall that $\sigma \in (0, \varsigma(s_1, s_2)]$, we have $0 \leq s_1 < s_2 < \bar{s} \leq 2$. Then by Proposition 2.1, we have

$$|u|_{2^*(s_2),s_2} \leq |u|^{\sigma}_{2^*(s_1),s_1}|u|^{1-\sigma}_{2^*(\bar{s}),\bar{s}}.$$

Recalling the Hardy-Sobolev inequality, we have $|u|_{2^*(\bar{s}),\bar{s}} \leq \mu_{\bar{s}}(\Omega)^{-\frac{1}{2}}\|u\|$. Hence, there exists a $C(\sigma) > 0$ such that

$$|u|_{2^*(s_2),s_2} \leq C(\sigma)\|u\|^{1-\sigma}|u|^{\sigma}_{2^*(s_1),s_1}.$$

□

Remark 2.1. *The above Corollary 2.1 and Corollary 2.2 are essentially the well known CKN inequality. However, based on the Proposition 2.1, our proofs are very concise. Moreover, the expressions of* (2.5) *and* (2.7) *are very convenient in our applications.*

3 Regularity, symmetry and decay estimation

In this section, we will study the regularity, symmetry and decay estimation about the positive solutions.

Lemma 3.1. *Assume that $0 < s_1 \leq s_2 < 2, 0 < u \in D_0^{1,2}(\mathbb{R}^N_+)$ and $|u|^{2^*(s_1)-1}/|x|^{s_1} \in L^q(B_1^+)$ for all $1 \leq q < q_1$, where $B_1^+ := B_1(0) \cap \mathbb{R}^N_+$. Then*

$$|u|^{2^*(s_2)-1}/|x|^{s_2} \in L^q(B_1^+) \quad \textit{for all } 1 \leq q < \frac{N(N + 2 - 2s_1)q_1}{N(N + 2 - 2s_2) + (s_2 - s_1)q_1}.$$

Further, if $|u|^{2^*(s_1)-1}/|x|^{s_1} \in L^q(B_1^+)$ for all $1 \leq q < \infty$, then we have

$$|u|^{2^*(s_2)-1}/|x|^{s_2} \in L^q(B_1^+) \quad \textit{for all } 1 \leq q < \frac{N(N + 2 - 2s_1)}{(N + 2)(s_2 - s_1)}.$$

Proof. When $q < \frac{N(N+2-2s_1)q_1}{N(N+2-2s_2)+(N+2)(s_2-s_1)q_1}$ and $0 < s_1 \leq s_2 < 2$, we see that

$$\frac{2^*(s_2) - 1}{2^*(s_1) - 1}\frac{q}{q_1} < 1 - \frac{s_2 q - \frac{2^*(s_2)-1}{2^*(s_1)-1}s_1 q}{N} \leq 1.$$

Then we can take some $\theta \in (0,1)$ such that

$$\frac{2^*(s_2)-1}{2^*(s_1)-1}\frac{q}{q_1} < \theta < 1 - \frac{s_2 q - \frac{2^*(s_2)-1}{2^*(s_1)-1}s_1 q}{N}.$$

Let

$$t = \frac{s_2 q - \frac{2^*(s_2)-1}{2^*(s_1)-1}q s_1}{1-\theta}, \qquad \tilde{q} = \frac{1}{\theta}\frac{2^*(s_2)-1}{2^*(s_1)-1}q.$$

Then by the choice of θ, we have $t < N$ and $\tilde{q} < q_1$. Hence, by the Hölder inequality, we have

$$\int_{B_1^+} \frac{u^{(2^*(s_2)-1)q}}{|x|^{s_2 q}}dx \le \Big(\int_{B_1^+} \frac{u^{(2^*(s_1)-1)\tilde{q}}}{|x|^{s_1\tilde{q}}}dx\Big)^{\theta}\Big(\int_{B_1^+}\frac{1}{|x|^t}\Big)^{1-\theta} < +\infty. \tag{3.1}$$

It is easy to see that $\dfrac{N(N+2-2s_1)q_1}{N(N+2-2s_2)+(N+2)(s_2-s_1)q_1}$ is increasing by q_1 and goes to $\dfrac{N(N+2-2s_1)}{(N+2)(s_2-s_1)}$ as $q_1 \to \infty$. $\square$

Lemma 3.2. *Assume that* $0 < s_2 \le s_1 < 2, 0 < u \in D_0^{1,2}(\mathbb{R}_+^N)$ *and* $|u|^{2^*(s_1)-1}/|x|^{s_1} \in L^q(B_1^+)$ *for all* $1 \le q < q_1$, *where* $B_1^+ := B_1(0) \cap \mathbb{R}_+^N$. *Then* $|u|^{2^*(s_2)-1}/|x|^{s_2} \in L^q(B_1^+)$ *for all* $1 \le q < \frac{2^*(s_1)-1}{2^*(s_2)-1}q_1$.

Proof. For any $1 \le q < \frac{2^*(s_1)-1}{2^*(s_2)-1}q_1$, we set $t = \frac{2^*(s_2)-1}{2^*(s_1)-1}s_1 q - s_2 q$ and $\tilde{q} = \frac{2^*(s_2)-1}{2^*(s_1)-1}q$. Then under the assumptions, it is easy to see that $t \ge 0$ and $1 < \tilde{q} < q_1$. Hence,

$$\begin{aligned}\int_{B_1^+} \frac{u^{(2^*(s_2)-1)q}}{|x|^{s_2 q}}dx &= \int_{B_1^+} \frac{u^{(2^*(s_1)-1)\tilde{q}}}{|x|^{s_1\tilde{q}}}|x|^t dx \\ &\le \int_{B_1^+} \frac{u^{(2^*(s_1)-1)\tilde{q}}}{|x|^{s_1\tilde{q}}}dx < +\infty. \end{aligned}\tag{3.2}$$

$\square$

We note that for some subset Ω_1 and some $q \ge 1$ such that

$$|u|^{2^*(s_2)-1}/|x|^{s_2}, |v|^{2^*(s_2)-1}/|x|^{s_2} \in L^q(\Omega_1),$$

then by Hölder inequality, we also have $\frac{|u|^{t_1}|v|^{t_2}}{|x|^{s_2}} \in L^q(\Omega_1)$ provided $0 < t_1, t_2 < 2^*(s_2)-1$ and $t_1 + t_2 = 2^*(s_2)-1$. Hence, we can obtain the following result:

Proposition 3.1. *Assume* $s_1, s_2 \in (0,2), \kappa > 0, \alpha > 1, \beta > 1, \alpha+\beta = 2^*(s_2)$, *then any positive solution* (u,v) *of*

$$\begin{cases} -\Delta u - \lambda\frac{|u|^{2^*(s_1)-2}u}{|x|^{s_1}} = \kappa\alpha\frac{1}{|x|^{s_2}}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \mathbb{R}_+^N, \\ -\Delta v - \mu\frac{|v|^{2^*(s_1)-2}v}{|x|^{s_1}} = \kappa\beta\frac{1}{|x|^{s_2}}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \mathbb{R}_+^N, \\ (u,v) \in \mathscr{D} := D_0^{1,2}(\mathbb{R}_+^N) \times D_0^{1,2}(\mathbb{R}_+^N), \end{cases}\tag{3.3}$$

satisfying the following properties:

(i) *if* $0 < \max\{s_1, s_2\} < \frac{N+2}{N}$*, then* $u, v \in C^2(\overline{\mathbb{R}^N_+})$*;*

(ii) *if* $\max\{s_1, s_2\} = \frac{N+2}{N}$*, then* $u, v \in C^{1,\gamma}(\mathbb{R}^N_+)$ *for all* $0 < \gamma < 1$*;*

(iii) *if* $\max\{s_1, s_2\} > \frac{N+2}{N}$*, then* $u, v \in C^{1,\gamma}(\mathbb{R}^N_+)$ *for all* $0 < \gamma < \frac{N(2-\max\{s_1,s_2\})}{N-2}$*.*

Proof. Indeed, it is enough to consider the regularity theorem at $0 \in \partial\mathbb{R}^N_+$. By [26, Lemma B.3], u, v are locally bounded. Let $B_1^+ := B_1(0) \cap \mathbb{R}^N_+$. We see that there exists some $C > 0$ such that

$$\begin{aligned} &|u(x)|^{2^*(s_1)-1}/|x|^{s_1} \le C|x|^{-s_1}, |v(x)|^{2^*(s_1)-1}/|x|^{s_1} \le C|x|^{-s_1}, \\ &|u(x)|^{2^*(s_2)-1}/|x|^{s_2} \le C|x|^{-s_2}, |v(x)|^{2^*(s_2)-1}/|x|^{s_2} \le C|x|^{-s_2} \end{aligned} \tag{3.4}$$

for $x \in B_1^+$. Hence

$$|u|^{2^*(s_1)-1}/|x|^{s_1}, |v|^{2^*(s_1)-1}/|x|^{s_1} \in L^q(B_1^+) \text{ for all } 1 \le q < \frac{N}{s_1}$$

and

$$\kappa\alpha\frac{1}{|x|^{s_2}}|u|^{\alpha-2}u|v|^{\beta},\ \kappa\beta\frac{1}{|x|^{s_2}}|u|^{\alpha}|v|^{\beta-2}v \in L^q(B_1^+) \text{ for all } 1 \le q < \frac{N}{s_2}.$$

Set $s_{max} := \max\{s_1, s_2\}$ and $s_{min} := \min\{s_1, s_2\}$. Then we have that $u, v \in W^{2,q}(B_1^+)$ for all $1 \le q < \frac{N}{s_{max}}$. Denote

$$\tau_u := \sup\{\tau :\ \sup_{B_1^+}(|u(x)|/|x|^{\tau}) < \infty, 0 < \tau < 1\},$$

$$\tau_v := \sup\{\tau :\ \sup_{B_1^+}(|v(x)|/|x|^{\tau}) < \infty, 0 < \tau < 1\}.$$

and

$$\tau_0 := \min\{\tau_u, \tau_v\}.$$

Step 1: We prove that $\tau_0 = 1$, i.e., $\tau_u = \tau_v = 1$.

Case 1: $s_{max} \le 1$. By the Sobolev embedding, we have $u, v \in C^{\tau}(\overline{B_1^+})$ for any $0 < \tau < 1$. Hence, $\tau_0 = 1$ in this case.

Case 2: $s_{max} > 1$. For this case, we have $u, v \in C^{\tau}(\overline{B_1^+})$ for all $0 < \tau < \min\{2 - s_{max}, 1\}$. Then by the definition, we have $2 - s_{max} \le \tau_0 \le 1$. For any $0 < \tau < \tau_0$, we have $|u(x)| \le C|x|^{\tau}$ and $|v(x)| \le C|x|^{\tau}$ for $x \in \overline{B_1^+}$, then for any $x \in B_1^+$, there exists some $C > 0$ such that

$$\begin{aligned} &|u(x)|^{2^*(s_1)-1}/|x|^{s_1} \le C|x|^{\left(2^*(s_1)-1\right)\tau - s_1}, \\ &|v(x)|^{2^*(s_1)-1}/|x|^{s_1} \le C|x|^{\left(2^*(s_1)-1\right)\tau - s_1}, \\ &|u(x)|^{2^*(s_2)-1}/|x|^{s_2} \le C|x|^{\left(2^*(s_2)-1\right)\tau - s_2}, \\ &|v(x)|^{2^*(s_2)-1}/|x|^{s_2} \le C|x|^{\left(2^*(s_2)-1\right)\tau - s_2}. \end{aligned} \tag{3.5}$$

Suppose $\tau_0 < 1$, then by (3.5), there must hold $\big(2^*(s_{max}) - 1\big)\tau_0 - s_{max} < 0$. Otherwise,

$$|u|^{2^*(s_{max})-1}/|x|^{s_{max}} \in L^q(B_1^+),\ |v|^{2^*(s_{max})-1}/|x|^{s_{max}} \in L^q(B_1^+)$$

for all $1 \le q < \infty$. On the other hand, by Lemma 3.2 and Hölder inequality, it is easy to prove that

$$\kappa\alpha\frac{1}{|x|^{s_2}}|u|^{\alpha-2}u|v|^{\beta}, \quad \kappa\beta\frac{1}{|x|^{s_2}}|u|^{\alpha}|v|^{\beta-2}v \in L^q(B_1^+) \text{ for all } 1 \le q < \infty.$$

It follows that $u \in W^{2,q}(B_{1/2}^+)$ for any $1 \le q < \infty$ and then by the Sobolev embedding again we have $\tau_0 = 1$, a contradiction. Therefore, $\big(2^*(s_{max}) - 1\big)\tau_0 - s_{max} < 0$ is proved and thus we have

$$|u|^{2^*(s_{max})-1}/|x|^{s_{max}}, \quad |v|^{2^*(s_{max})-1}/|x|^{s_{max}} \in L^q(B_1^+)$$

for all $1 \le q < \dfrac{N}{s_{max} - \big(2^*(s_{max}) - 1\big)\tau_0}$.

Subcase 2.1: If $s_{min} \le 1$ or $\big(2^*(s_{min}) - 1\big)\tau_0 - s_{min} \ge 0$, we have

$$|u|^{2^*(s_{min})-1}/|x|^{s_{min}},\ |v|^{2^*(s_{min})-1}/|x|^{s_{min}} \in L^q(B_1^+)$$

for all $1 \le q < N$. We claim that $s_{max} - \big(2^*(s_{max}) - 1\big)\tau_0 > 1$. If not, we see that $u, v \in W^{2,q}(B_1^+)$ for all $1 \le q < N$, and then by Sobolev embedding, we obtain that $\tau_0 = 1$, a contradiction. Hence, we have $u, v \in W^{2,q}(B_1^+)$ for all $1 \le q < \frac{N}{s_{max}-\big(2^*(s_{max})-1\big)\tau_0}$, and by the Sobolev embedding again, we have $u, v \in C^{\tau}(\overline{B_{1/2}^+})$ for all $0 < \tau < \min\{2 - \big[s_{max} - \big(2^*(s_{max}) - 1\big)\tau_0\big], 1\}$. Then by the definition of τ_0, we should have

$$2 - \big[s_{max} - \big(2^*(s_{max}) - 1\big)\tau_0\big] \le \tau_0$$

which implies that

$$2 - s_{max} + \big(2^*(s_{max}) - 2\big)\tau_0 \le 0.$$

But $s_{max} < 2, 2^*(s_{max}) > 2, \tau_0 > 0$, a contradiction again.

Subcase 2.2: If $1 < s_{min} \le s_{max} < 2$ and $\big(2^*(s_{min}) - 1\big)\tau_0 - s_{min} < 0$, by Lemma 3.2 again,

$$|u|^{2^*(s_{min})-1}/|x|^{s_{min}}, \quad |v|^{2^*(s_{min})-1}/|x|^{s_{min}} \in L^q(B_1^+)$$

for all

$$1 \le q < \frac{2^*(s_{max}) - 1}{2^*(s_{min}) - 1}\frac{N}{s_{max} - \big(2^*(s_{max}) - 1\big)\tau_0}.$$

On the other hand, by the definition of τ_0, we have that

$$|u|^{2^*(s_{min})-1}/|x|^{s_{min}},\ |v|^{2^*(s_{min})-1}/|x|^{s_{min}} \in L^q(B_1^+)$$

for all $1 \le q < \frac{N}{s_{min}-\left(2^*(s_{min})-1\right)\tau_0}$. Thus,

$$|u|^{2^*(s_{min})-1}/|x|^{s_{min}}, \qquad |v|^{2^*(s_{min})-1}/|x|^{s_{min}} \in L^q(B_1^+)$$

for all

$$1 \le q < \max\left\{\frac{N}{s_{min}-\left(2^*(s_{min})-1\right)\tau_0}, \frac{2^*(s_{max})-1}{2^*(s_{min})-1}\frac{N}{s_{max}-\left(2^*(s_{max})-1\right)\tau_0}\right\}.$$

Noting that

$$\begin{aligned}
&\frac{2^*(s_{max})-1}{2^*(s_{min})-1}\frac{N}{s_{max}-\left(2^*(s_{max})-1\right)\tau_0} \\
&\le \frac{N}{s_{max}-\left(2^*(s_{max})-1\right)\tau_0} \\
&\le \frac{N}{s_{min}-\left(2^*(s_{min})-1\right)\tau_0},
\end{aligned}$$

we have $|u|^{2^*(s_{min})-1}/|x|^{s_{min}}$, $|v|^{2^*(s_{min})-1}/|x|^{s_{min}} \in L^q(B_1^+)$ for all $1 \le q < \frac{N}{s_{min}-\left(2^*(s_{min})-1\right)\tau_0}$ and it follows that

$$u, v \in W^{2,q}(B_1^+) \text{ for } 1 \le q < \frac{N}{s_{max}-\left(2^*(s_{max})-1\right)\tau_0}.$$

Then apply the similar arguments as that in the subcase 2.1, we can deduce a contradiction. Hence, $\tau_0 = 1$ is proved and then $\tau_u = \tau_v = 1$, i.e., for any $0 < \tau < 1$,

$$\begin{aligned}
|u(x)|^{2^*(s_1)-1}/|x|^{s_1} &\le C|x|^{\left(2^*(s_1)-1\right)\tau - s_1}, \\
|v(x)|^{2^*(s_1)-1}/|x|^{s_1} &\le C|x|^{\left(2^*(s_1)-1\right)\tau - s_1}, \\
|u(x)|^{2^*(s_2)-1}/|x|^{s_2} &\le C|x|^{\left(2^*(s_2)-1\right)\tau - s_2}, \\
|v(x)|^{2^*(s_2)-1}/|x|^{s_2} &\le C|x|^{\left(2^*(s_2)-1\right)\tau - s_2}. \qquad (3.6)
\end{aligned}$$

Step 2: We prove that $u, v \in W^{2,q}(B_1^+)$ **for all**

$$1 \le q < \begin{cases} \infty & \textbf{if } 2^*(s_{max})-1-s_{max} \ge 0 \\ \frac{N}{1+s_{max}-2^*(s_{max})} & \textbf{if } 2^*(s_{max})-1-s_{max} < 0 \end{cases}.$$

We divide the proof in two cases.

Case 1: $2^*(s_{max})-1-s_{max} \ge 0$, *i.e.,* $s_{max} \le \frac{N+2}{N}$. By taking τ close to 1, we see that

$$|u|^{2^*(s_1)-1}/|x|^{s_1}, |v|^{2^*(s_1)-1}/|x|^{s_1}, |u|^{2^*(s_2)-1}/|x|^{s_2}, |v|^{2^*(s_2)-1}/|x|^{s_2} \in L^q(B_1^+)$$

for all $1 < q < \infty$. Meanwhile, by the Hölder inequality,

$$\kappa\alpha\frac{1}{|x|^{s_2}}|u|^{\alpha-2}u|v|^{\beta},\ \kappa\beta\frac{1}{|x|^{s_2}}|u|^{\alpha}|v|^{\beta-2}v \in L^q(B_1^+) \text{ for all } 1 \le q < \infty.$$

Hence, $u, v \in W^{2,q}(B_{\frac{1}{2}}^+)$ for all $1 \le q < \infty$.

Case 2: $2^*(s_{max}) - 1 - s_{max} < 0$, *i.e.,* $\frac{N+2}{N} < s_{max} < 2$. In this case, we have

$$|u|^{2^*(s_{max})-1}/|x|^{s_{max}}, |v|^{2^*(s_{max})-1}/|x|^{s_{max}} \in L^q(B_1^+)$$

for all

$$1 < q < \frac{N}{1 + s_{max} - 2^*(s_{max})}.$$

If $2^*(s_{min}) - 1 - s_{min} \ge 0$, then we see that

$$|u|^{2^*(s_{min})-1}/|x|^{s_{min}}, |v|^{2^*(s_{min})-1}/|x|^{s_{min}} \in L^q(B_1^+) \text{ for all } 1 < q < \infty.$$

Hence, $u, v \in W^{2,q}(B_1^+)$ for all $1 \le q < \frac{N}{1+s_{max}-2^*(s_{max})}$.

If $2^*(s_{min}) - 1 - s_{min} < 0$, we must have

$$|u|^{2^*(s_{min})-1}/|x|^{s_{min}}, |v|^{2^*(s_{min})-1}/|x|^{s_{min}} \in L^q(B_1^+)$$

for all $1 < q < \frac{N}{1+s_{min}-2^*(s_{min})}$. Since

$$\frac{N}{1 + s_{min} - 2^*(s_{min})} \ge \frac{N}{1 + s_{max} - 2^*(s_{max})},$$

we also obtain that $u, v \in W^{2,q}(B_1^+)$ for all $1 \le q < \frac{N}{1+s_{max}-2^*(s_{max})}$.

Step 3: By the Sobolev embedding theorem,

$$u, v \in C^{1,\gamma}(\overline{B_{1/2}^+}) \text{ for all } 0 < \gamma < 1 \text{ if } s_{max} \le \frac{N+2}{N}.$$

In particular, in the case $s_{max} < \frac{N+2}{N}$, there exists $q_0 > N$ such that

$$\begin{aligned}
&\|u\|_{W^{3,q_0}(B_{1/2}^+)} \\
\le &C\Big(1 + \|\frac{u^{2^*(s_1)-2}\nabla u}{|x|^{s_1}}\|_{L^{q_0}(B_1^+)} + \|\frac{u^{2^*(s_1)-1}}{|x|^{s_1+1}}\|_{L^{q_0}(B_1^+)} + \|\frac{u^{\alpha-2}v^{\beta}\nabla u}{|x|^{s_2}}\|_{L^{q_0}(B_1^+)} \\
&+ \|\frac{u^{\alpha-1}v^{\beta-1}\nabla v}{|x|^{s_2}}\|_{L^{q_0}(B_1^+)} + \|\frac{u^{\alpha-1}v^{\beta}}{|x|^{s_2+1}}\|_{L^{q_0}(B_1^+)}\Big) < \infty.
\end{aligned}$$

Thus, we obtain that $u \in C^2(\overline{B_{1/2}^+})$. Similarly, we can also prove that $v \in C^2(\overline{B_{1/2}^+})$. If $s_{max} > \frac{N}{N+2}$, note that $\frac{N}{1+s_{max}-2^*(s_{max})} > N$, by taking τ close to 1, we have $u, v \in C^{1,\gamma}(\overline{B_{1/2}^+})$ for all $0 < \gamma < 1 - [1 + s_{max} - 2^*(s_{max})] = \frac{N(2-s_{max})}{N-2}$. $\square$

Proposition 3.2. *Assume that* $s_1, s_2 \in (0,2), \kappa > 0, \alpha > 1, \beta > 1, \alpha+\beta = 2^*(s_2)$. *Let* (u,v) *be a positive solution of* (3.3)*, then there exists a constant* C *such that*

$$|u(x)|, |v(x)| \le C(1+|x|^{-(N-1)}); \quad |\nabla u(x)|, |\nabla v(x)| \le C|x|^{-N}.$$

Proof. Recalling the Kelvin transformation:

$$u^*(y) := |y|^{-(N-2)} u(\frac{y}{|y|^2}), \;\; v^*(y) := |y|^{-(N-2)} v(\frac{y}{|y|^2}). \tag{3.7}$$

It is well known that

$$\Delta u^*(y) = \frac{1}{|y|^{N+2}} (\Delta u)(\frac{y}{|y|^2}) \text{ and } \Delta v^*(y) = \frac{1}{|y|^{N+2}} (\Delta v)(\frac{y}{|y|^2}). \tag{3.8}$$

Hence, a direct computation shows that $\big(u^*, v^*\big)$ is also a positive solution to the same equation.

By Proposition 3.1, we see that $u^*, v^* \in C^{1,\gamma}(\overline{\mathbb{R}^N_+})$ for some $\gamma > 0$. Then $|u^*(y)|, |v^*(y)| \le C|y|$ for $y \in B_1^+$. Going back to (u,v), we see that $|u(y)|, |v(y)| \le C|y|^{-(N-1)}$ for $y \in \mathbb{R}^N_+$. Finally, it is standard to apply the gradient estimate, we obtain that $|\nabla u(y)|, |\nabla v(y)| \le C|y|^{-N}$ for $y \in \mathbb{R}^N_+$. ☐

Remark 3.1. *Checking the proofs of Lemmas 3.1-3.2 and Propositions 3.1-3.2, their conclusions are valid for general cone* Ω. *A little difference is that when* $\Omega = \mathbb{R}^N$*, the decay estimation is*

$$|u(x)|, |v(x)| \le C(1+|x|^{-N}); \quad |\nabla u(x)|, |\nabla v(x)| \le C|x|^{-N-1}.$$

Proposition 3.3. *Assume that* $s_1, s_2 \in (0,2), \kappa > 0, \alpha > 1, \beta > 1, \alpha+\beta = 2^*(s_2)$. *Let* (u,v) *be a positive solution of* (3.3)*. Then we have that* $u\circ\sigma = u, v\circ\sigma = v$ *for all isometry of* $\mathbb{R}^N$ *such that* $\sigma(\mathbb{R}^N_+) = \mathbb{R}^N_+$*. In particular,* $\big(u(x', x_N), v(x', x_N)\big)$ *is axially symmetric with respect to the* x_N-*axis, i.e.,* $u(x', x_N) = u(|x'|, x_N)$ *and* $v(x', x_N) = v(|x'|, x_N)$.

Proof. We prove the result by the well-known method of moving planes. Denote by $\overrightarrow{e_N}$ the N^{th} vector of the canonical basis of $\mathbb{R}^N$ and consider the open ball $D := B_{\frac{1}{2}}(\frac{1}{2}\overrightarrow{e_N})$. Set

$$\begin{cases} \varphi(x) := |x|^{2-N} u(-\overrightarrow{e_N} + \frac{x}{|x|^2}), \\ \psi(x) := |x|^{2-N} v(-\overrightarrow{e_N} + \frac{x}{|x|^2}) \end{cases} \tag{3.9}$$

for all $x \in \bar{D}\backslash\{0\}$ and $\varphi(0) = \psi(0) = 0$. By Proposition 3.1, $\varphi(x), \psi(x) \in C^2(D) \cap C^1(\bar{D}\backslash\{0\})$. We note that it is easy to see that $\varphi(x) > 0, \psi(x) > 0$ in D and $\varphi(x) = \psi(x) = 0$ on $\partial D\backslash\{0\}$. On the other hand, by Proposition 3.2, there exists $C > 0$ such that

$$\varphi(x) \le C|x|, \psi(x) \le C|x| \text{ for all } x \in \bar{D}\backslash\{0\}. \tag{3.10}$$

Since $\varphi(0)=\psi(0)=0$, we have that $\varphi(x),\psi(x)\in C^0(\bar{D})$. By a direct computation, $\big(\varphi(x),\psi(x)\big)$ satisfies the following eqaution

$$\begin{cases}-\Delta\varphi-\lambda\frac{\varphi^{2^*(s_1)-1}}{\left|x-|x|^2\overrightarrow{e_N}\right|^{s_1}}=\kappa\alpha\frac{\varphi^{\alpha-1}\psi^{\beta}}{\left|x-|x|^2\overrightarrow{e_N}\right|^{s_2}}\\ -\Delta\psi-\mu\frac{\psi^{2^*(s_1)-1}}{\left|x-|x|^2\overrightarrow{e_N}\right|^{s_1}}=\kappa\beta\frac{\varphi^{\alpha}\psi^{\beta-1}}{\left|x-|x|^2\overrightarrow{e_N}\right|^{s_2}}\end{cases}\quad \text{in } D. \tag{3.11}$$

Noting that

$$\left|x-|x|^2\overrightarrow{e_N}\right|=\left|x\right|\left|x-\overrightarrow{e_N}\right|, \tag{3.12}$$

we have

$$\begin{cases}-\Delta\varphi-\lambda\frac{\varphi^{2^*(s_1)-1}}{\left|x\right|^{s_1}\left|x-\overrightarrow{e_N}\right|^{s_1}}=\kappa\alpha\frac{\varphi^{\alpha-1}\psi^{\beta}}{\left|x\right|^{s_2}\left|x-\overrightarrow{e_N}\right|^{s_2}}\\ -\Delta\psi-\mu\frac{\psi^{2^*(s_1)-1}}{\left|x\right|^{s_1}\left|x-\overrightarrow{e_N}\right|^{s_1}}=\kappa\beta\frac{\varphi^{\alpha}\psi^{\beta-1}}{\left|x\right|^{s_1}\left|x-\overrightarrow{e_N}\right|^{s_1}}\end{cases}\quad \text{in } D. \tag{3.13}$$

Since $\overrightarrow{e_N}\in\partial D\backslash\{0\}$ and $\varphi(x),\psi(x)\in C^1(\bar{D}\backslash\{0\})\cap C^0(\bar{D})$, there exists $C>0$ such that

$$\varphi(x)\leq C|x-\overrightarrow{e_N}|,\psi(x)\leq C|x-\overrightarrow{e_N}| \text{ for all } x\in\bar{D}. \tag{3.14}$$

Noting that $2^*(s_i)-1-s_i>-N$ for $i=1,2$, then by (3.10), (3.13), (3.14) and the standard elliptic theory, we obtain that $\varphi(x),\psi(x)\in C^1(\bar{D})$. By $\varphi(x)>0,\psi(x)>0$ in D, we obtain that $\frac{\partial\varphi}{\partial\nu}<0,\frac{\partial\psi}{\partial\nu}<0$ on ∂D, where ν denotes the outward unit normal to D at $x\in\partial D$.

For any $\eta\geq 0$ and any $x=(x_1,x')\in\mathbb{R}^N$, where $x'=(x_2,\cdots,x_N)\in\mathbb{R}^{N-1}$, we let

$$x_\eta=(2\eta-x_1,x') \text{ and } D_\eta:=\{x\in D\big|x_\eta\in D\}. \tag{3.15}$$

We say that (P_η) holds iff

$$D_\eta\neq\emptyset \text{ and } \varphi(x)\geq\varphi(x_\eta),\psi(x)\geq\psi(x_\eta) \text{ for all } x\in D_\eta \text{ such that } x_1\leq\eta.$$

Step 1: We shall prove that (P_η) holds if $\eta<\frac{1}{2}$ and close to $\frac{1}{2}$ sufficiently.

Indeed, it is easily to follow the Hopf's Lemma (see the arguments above) that there exists $\varepsilon_0>0$ such that (P_η) holds for $\eta\in(\frac{1}{2}-\varepsilon_0,\frac{1}{2})$. Now, we let

$$\sigma:=\min\{\eta\geq 0;(P_\delta)\text{ holds for all }\delta\in(\eta,\frac{1}{2})\}. \tag{3.16}$$

Step 2: We shall prove that $\sigma=0$.

We prove it by way of negation. Assume that $\sigma>0$, then we see that $D_\sigma\neq\emptyset$ and that (P_σ) holds. Now, we set

$$\hat{\varphi}(x):=\varphi(x)-\varphi(x_\sigma) \text{ and } \hat{\psi}(x):=\psi(x)-\psi(x_\sigma). \tag{3.17}$$

Then we have

$$
\begin{aligned}
-\Delta\hat{\varphi} =&[-\Delta\varphi(x)]-[-\Delta\varphi(x_\sigma)]\\
=&\lambda\frac{\big(\varphi(x)\big)^{2^*(s_1)-1}}{|x-|x|^2\overrightarrow{e_N}|^{s_1}}+\kappa\alpha\frac{\big(\varphi(x)\big)^{\alpha-1}\big(\psi(x)\big)^{\beta}}{|x-|x|^2\overrightarrow{e_N}|^{s_2}}\\
&-\lambda\frac{\big(\varphi(x_\sigma)\big)^{2^*(s_1)-1}}{|x_\sigma-|x_\sigma|^2\overrightarrow{e_N}|^{s_1}}-\kappa\alpha\frac{\big(\varphi(x_\sigma)\big)^{\alpha-1}\big(\psi(x_\sigma)\big)^{\beta}}{|x_\sigma-|x_\sigma|^2\overrightarrow{e_N}|^{s_2}}\\
\geq&\lambda\big(\varphi(x_\sigma)\big)^{2^*(s_1)-1}\Big[\frac{1}{|x-|x|^2\overrightarrow{e_N}|^{s_1}}-\frac{1}{|x_\sigma-|x_\sigma|^2\overrightarrow{e_N}|^{s_1}}\Big]\\
&+\kappa\alpha\big(\varphi(x_\sigma)\big)^{\alpha-1}\big(\psi(x)\big)^{\beta}\Big[\frac{1}{|x-|x|^2\overrightarrow{e_N}|^{s_2}}-\frac{1}{|x_\sigma-|x_\sigma|^2\overrightarrow{e_N}|^{s_2}}\Big]
\end{aligned}
\tag{3.18}
$$

for all $x\in D_\sigma\cap\{x_1<\sigma\}$. Noting that

$$
\begin{aligned}
&\big|x_\sigma+|x_\sigma|^2\overrightarrow{e_N}\big|^2-\big|x+|x|^2\overrightarrow{e_N}\big|^2\\
&=(|x_\sigma|^2-|x|^2)(1+|x_\sigma|^2+|x|^2+2x_N)\\
&=4\sigma(\sigma-x_1)(1+|x_\sigma|^2+|x|^2+2x_N),
\end{aligned}
\tag{3.19}
$$

we obtain that

$$-\Delta\hat{\varphi}(x)>0 \text{ for all } x\in D_\sigma\cap\{x_1<\sigma\}. \tag{3.20}$$

Similarly, we also have

$$-\Delta\hat{\psi}(x)>0 \text{ for all } x\in D_\sigma\cap\{x_1<\sigma\}. \tag{3.21}$$

Then by the Hopf's Lemma and the strong comparison principle, we have

$$\hat{\varphi},\hat{\psi}>0 \text{ in } D_\sigma\cap\{x_1<\sigma\} \text{ and } \frac{\partial\hat{\varphi}}{\partial\nu},\frac{\partial\hat{\psi}}{\partial\nu}<0 \text{ on } D_\sigma\cap\{x_1=\sigma\}. \tag{3.22}$$

Here we use the assumption $\sigma>0$. By definition, there exists a subsequence $\{\sigma_i\}_{i\in\mathbb{N}}\subset\mathbb{R}^+$ and a sequence $\{x^i\}_{i\in\mathbb{N}}\subset D$ such that $\sigma_i<\sigma, x^i\in D_{\sigma_i}, (x^i)_1<\sigma_i, \lim\limits_{i\to\infty}\sigma_i=\sigma$ and

$$\varphi(x^i)<\varphi((x^i)_{\sigma_i}) \text{ or } \psi(x^i)<\psi((x^i)_{\sigma_i}). \tag{3.23}$$

Up to a subsequence, we may assume that $\varphi(x^i)<\varphi((x^i)_{\sigma_i})$ without loss of generality. Since $\{x^i\}_{i\in\mathbb{N}}$ is bounded, going to a subsequence again, we assume that $\lim\limits_{i\to\infty}x_i=x\in\overline{D_\sigma}\cap\{x_1\leq\sigma\}$ due to the choice of $\{x^i\}$. Then we have $\varphi(x)\leq\varphi(x_\sigma)$, i.e., $\hat{\varphi}(x)\leq 0$. Combining with (3.23), we obtain that $\hat{\varphi}(x)=0$ and then $x\in\partial\big(D_\sigma\cap\{x_1<\sigma\}\big)$.

Case 1: If $x\in\partial D$, then $\varphi(x)=0$. It follows that $\varphi(x_\sigma)=0$. Since $x_\sigma\in D$ and $\varphi>0$ in D, we also have $x_\sigma\in\partial D$. We say that $x=x_\sigma$. If not, x and x_σ are symmetric with respect to the hyperplane $x_1=\sigma$. This is impossible since that D is a

ball, $\sigma > 0$ and $x, x_\sigma \in \partial D$. Now recalling that $\varphi \in C^1$, by the mean value theorem, there exists a sequence $\tau_i \in \big((x^i)_1, 2\sigma_i - (x^i)_1\big)$ such that

$$\varphi(x^i) - \varphi((x^i)_{\sigma_i}) = 2\partial_1\varphi(\tau_i, (x')^i)\ \big((x^i)_1 - \sigma_i\big). \tag{3.24}$$

Using the facts $(x^i)_1 < \sigma_i$ and $\varphi(x^i) < \varphi\big((x^i)_{\sigma_i}\big)$, we let i go to infinity and then obtain that

$$\partial_1\varphi(x) \geq 0. \tag{3.25}$$

On the other hand,

$$\begin{aligned}\partial_1\varphi(x) =& \frac{\partial\varphi}{\partial\nu}\langle\nu(x), \overrightarrow{e_1}\rangle \\ =& \frac{\sigma}{|x - \frac{1}{2}\overrightarrow{e_N}|}\frac{\partial\varphi}{\partial\nu} < 0,\end{aligned} \tag{3.26}$$

a contradiction. Here we use the assumption $\sigma > 0$ again and the fact $x_1 = \sigma$ since $x = x_\sigma$.

Case 2: If $x \in D$, since $\varphi(x) = \varphi(x_\sigma) > 0$, we have $x_\sigma \in D$. Since $x \in \partial\big(D_\sigma \cap \{x_1 < \sigma\}\big)$, we have $x \in D \cap \{x_1 = \sigma\}$. Then apply the similar arguments in Case 1, we can also obtain that $\partial_1\varphi(x) \geq 0$. On the other hand, by (3.23), we have $2\partial_1\varphi(x) = \partial_1\hat{\varphi}(x) < 0$, also a contradiction.

Hence, $\sigma = 0$ is proved.

Step 3: By Step 2, $\sigma = 0$. Hence, we have

$$\varphi(x_1, x') \geq \varphi(-x_1, x'), \psi(x_1, x') \geq \psi(-x_1, x') \text{ for all } x \in D_0 = D. \tag{3.27}$$

Apply the same argument one can obtain the reverse inequality. Thus,

$$\varphi(x_1, x') = \varphi(-x_1, x'), \psi(x_1, x') = \psi(-x_1, x') \text{ for all } x \in D. \tag{3.28}$$

Hence, φ and ψ are symmetric with respect to the hyperplane $\{x_1 = 0\}$. Noting that the arguments above are valid for any hyperplane containing $\overrightarrow{e_N}$. By going back to (u, v), we obtain the conclusions of this proposition. $\square$

Remark 3.2. **(Open problem)** *The Proposition 3.3 is established under the assumption that $\Omega = \mathbb{R}^N_+$. But so far we do not know whether the Proposition 3.3 is true for general cone Ω. It remains an open problem.*

4 Nehari manifold

In this section, we study the Nehari manifold corresponding to the following equation:

$$\begin{cases}-\Delta u - \lambda\frac{|u|^{2^*(s_1)-2}u}{|x|^{s_1}} = \kappa\alpha\frac{1}{|x|^{s_2}}|u|^{\alpha-2}u|v|^\beta & \text{in } \mathbb{R}^N_+, \\ -\Delta v - \mu\frac{|v|^{2^*(s_1)-2}v}{|x|^{s_1}} = \kappa\beta\frac{1}{|x|^{s_2}}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \mathbb{R}^N_+, \\ (u, v) \in \mathscr{D} := D^{1,2}_0(\mathbb{R}^N_+) \times D^{1,2}_0(\mathbb{R}^N_+),\end{cases} \tag{4.1}$$

For $(u,v)\in\mathscr{D}$, we define the norm

$$\|(u,v)\|_{\mathscr{D}}=\left(\|u\|^2+\|v\|^2\right)^{\frac{1}{2}},$$

where $\|u\|:=\left(\int_{\mathbb{R}^N_+}|\nabla u|^2dx\right)^{\frac{1}{2}}$ for $u\in D_0^{1,2}(\mathbb{R}^N_+)$. A pair of function (u,v) is said to be a weak solution of (4.1) if and only if

$$\begin{aligned}&\int_{\mathbb{R}^N_+}\nabla u\nabla\varphi_1+\nabla v\nabla\varphi_2dx-\lambda\int_{\mathbb{R}^N_+}\frac{|u|^{2^*(s_1)-2}u\varphi_1}{|x|^{s_1}}dx-\mu\int_{\mathbb{R}^N_+}\frac{|v|^{2^*(s_1)-2}v\varphi_2}{|x|^{s_1}}dx\\&-\kappa\alpha\int_{\mathbb{R}^N_+}\frac{|u|^{\alpha-2}u|v|^{\beta}\varphi_1}{|x|^{s_2}}dx-\kappa\beta\int_{\mathbb{R}^N_+}\frac{|u|^{\alpha}|v|^{\beta-2}v\varphi_2}{|x|^{s_2}}dx=0\text{ for all }(\varphi_1,\varphi_2)\in\mathscr{D}.\end{aligned}$$

The corresponding energy functional of problem (4.1) is defined as

$$\Phi(u,v)=\frac{1}{2}a(u,v)-\frac{1}{2^*(s_1)}b(u,v)-\kappa c(u,v)$$

for all $(u,v)\in\mathscr{D}$, where

$$\begin{cases}a(u,v):=\|(u,v)\|^2_{\mathscr{D}},\\ b(u,v):=\lambda\int_{\mathbb{R}^N_+}\frac{|u|^{2^*(s_1)}}{|x|^{s_1}}dx+\mu\int_{\mathbb{R}^N_+}\frac{|v|^{2^*(s_1)}}{|x|^{s_1}}dx,\\ c(u,v):=\int_{\mathbb{R}_+N}\frac{|u|^{\alpha}|v|^{\beta}}{|x|^{s_2}}dx.\end{cases}\tag{4.2}$$

We consider the corresponding Nehari manifold

$$\mathcal{N}:=\{(u,v)\in\mathscr{D}\backslash(0,0)|J(u,v)=0\}$$

where

$$\begin{aligned}J(u,v):=&\langle\Phi'(u,v),(u,v)\rangle\\=&a(u,v)-b(u,v)-\kappa(\alpha+\beta)c(u,v)\end{aligned}$$

and $\Phi'(u,v)$ denotes the Fréchet derivative of Φ at (u,v) and $\langle\cdot,\cdot\rangle$ is the duality product between $\mathscr{D}$ and its dual space $\mathscr{D}^*$.

Lemma 4.1. *Assume $s_1,s_2\in(0,2),\lambda,\mu\in(0,+\infty),\alpha>1,\beta>1$ and $\alpha+\beta=2^*(s_2)$. Then for any $(u,v)\in\mathscr{D}\backslash\{(0,0)\}$, there exists a unique $t=t_{(u,v)}>0$ such that $t(u,v)=(tu,tv)\in\mathcal{N}$ if one of the following assumptions is satisfied:*

(i) $\kappa>0$.

(ii) $\kappa<0$ *and* $s_2>s_1$.

(iii) $s_2=s_1$ *and* $\kappa<0$ *with* $|\kappa|$ *small enough.*

Moreover, $\mathcal{N}$ is closed and bounded away from 0.

Proof. For any $(u,v) \in \mathscr{D}$, we use the notations defined by (4.2) and we will write them as a, b, c for simplicity. It is easy to see that for any $(u,v) \neq (0,0)$, we have $a > 0, b > 0, c \geq 0$. For any $(u,v) \in \mathscr{D}\backslash\{(0,0)\}$ and $t > 0$, we have

$$\Phi(tu,tv) = \frac{1}{2}at^2 - \frac{1}{2^*(s_1)}bt^{2^*(s_1)} - \kappa ct^{2^*(s_2)}. \tag{4.3}$$

Denote $\frac{d\Phi(tu,tv)}{dt} := -tg(t)$, where

$$g(t) = bt^{2^*(s_1)-2} + \kappa 2^*(s_2)ct^{2^*(s_2)-2} - a.$$

For the cases of (i) and (ii), it is easy to see that $g(+\infty) = +\infty$ and $g(0) = -a < 0$. Also for the case (iii), by the Young inequality, one can prove that there exists some $C > 0$ such that $c(u,v) \leq Cb(u,v)$ for all $(u,v) \in \mathscr{D}$. Thus, for the case of $s_2 = s_1$, if $\kappa < 0$ with $|\kappa|$ small enough, we obtain that

$$b(u,v) + 2^*(s_2)\kappa c(u,v) > 0 \text{ for all } (u,v) \in \mathscr{D}\backslash\{(0,0)\}. \tag{4.4}$$

Hence, we also have $g(+\infty) = +\infty$ and $g(0) = -a < 0$.

Thus, we obtain that there exists some $t > 0$ such that $g(t) = 0$ due to the continuity of $g(t)$. It follows that $tu \in \mathcal{N}$. By the Hardy-Sobolev inequality and the Young inequality, there exists some $C > 0$ such that

$$b(u,v) \leq C\|(u,v)\|_{\mathscr{D}}^{2^*(s_1)}, \quad c(u,v) \leq C\|(u,v)\|_{\mathscr{D}}^{2^*(s_2)}.$$

Let $(u,v) \in \mathcal{N}$, since $2^*(s_1) > 2, i = 1,2$, we have

$$a = b + \kappa(2^*(s_2))c \leq C\Big(a^{\frac{2^*(s_1)}{2}} + a^{\frac{2^*(s_2)}{2}}\Big),$$

which implies that there exists some $\delta_0 > 0$ such that

$$\|(u,v)\|_{\mathscr{D}} = a^{\frac{1}{2}} \geq \delta_0 \text{ for all } (u,v) \in \mathcal{N}. \tag{4.5}$$

Thus, $\mathcal{N}$ is bounded away from $(0,0)$ and obviously, $\mathcal{N}$ is closed.

For any $(u,v) \neq (0,0)$, let $t_0 := \inf\{t|g(t) = 0, t > 0\}$. Then we see that $t_0 > 0$ and $g(t_0) = 0$. Without loss of generality, we may assume that $t_0 = 1$, that is, $g(t) < 0$ for $0 < t < 1$ and $g(1) = 0 = b + \kappa 2^*(s_2)c - a$. We note that

$$g'(t) = \big(2^*(s_1) - 2\big)bt^{2^*(s_1)-3} + \kappa 2^*(s_2)\big(2^*(s_2) - 2\big)ct^{2^*(s_2)-3}.$$

(i) If $\kappa > 0$, then $g'(t) > 0$ for all $t > 0$.
(ii) If $\kappa < 0, s_2 > s_1$, recalling that $0 = b + \kappa 2^*(s_2)c - a$, we have

$$\begin{aligned} g'(t) \equiv& \big(2^*(s_1) - 2\big)bt^{2^*(s_1)-3} + \kappa 2^*(s_2)\big(2^*(s_2) - 2\big)ct^{2^*(s_2)-3} \\ =& \Big[\big(2^*(s_1) - 2\big)bt^{2^*(s_1)-2^*(s_2)} + \kappa 2^*(s_2)\big(2^*(s_2) - 2\big)c\Big]t^{2^*(s_2)-3} \\ =&: h(t)t^{2^*(s_2)-3}, \end{aligned}$$

where

$$h(t) := \big(2^*(s_1)-2\big)bt^{2^*(s_1)-2^*(s_2)} + \kappa 2^*(s_2)\big(2^*(s_2)-2\big)c.$$

When $t>1$, we have

$$\begin{aligned} h(t) >& \big(2^*(s_1)-2\big)b + \kappa 2^*(s_2)\big(2^*(s_2)-2\big)c \\ =& \big(2^*(s_1)-2\big)\big(a-\kappa 2^*(s_2)c\big) + \kappa 2^*(s_2)\big(2^*(s_2)-2\big)c \\ =& \big(2^*(s_1)-2\big)a - \kappa 2^*(s_2)(2^*(s_1)-2^*(s_2))c \\ >& 0. \end{aligned}$$

Hence, $g'(t)>0$ for all $t>1$.
(iii) If $\kappa<0, s_2=s_1$, similar to the arguments as case (ii) above, we know

$$h(t) = \big(2^*(s_1)-2\big)\big(b+2^*(s_1)\kappa c\big) > 0$$

when κ is small enough by (4.4).

The arguments above imply that $g(t)>0$ for $t>1$. Hence, $t=1$ is the unique solution of $g(t)=0$. It follows that for any $(u,v)\neq(0,0)$, there exists a unique $t_{(u,v)}>0$ such that $t_{(u,v)}(u,v)\in\mathcal{N}$ and

$$\Phi(t_{(u,v)}u, t_{(u,v)}v) = \max_{t>0}\Phi(tu,tv).$$

□

Lemma 4.2. *Under the assumptions of Lemma 4.1, any $(PS)_m$ sequence of $\Phi(u,v)$ i.e.,*

$$\begin{cases} \Phi(u_n,v_n)\to m \\ \Phi'(u_n,v_n)\to 0 \text{ in } \mathscr{D}^* \end{cases}$$

is bounded in $\mathscr{D}$.

Proof. Let $\{(u_n,v_n)\}\subset\mathscr{D}$ be a $(PS)_m$ sequence of $\Phi(u,v)$. We tend to use the previous marks a,b,c and denote $a(u_n,v_n), b(u_n,v_n), c(u_n,v_n)$ by a_n,b_n,c_n for the simplicity. Then we have

$$\Phi(u_n,v_n) = \frac{1}{2}a_n - \frac{1}{2^*(s_1)}b_n - \kappa c_n = m + o(1) \tag{4.6}$$

and

$$J(u_n,v_n) = a_n - b_n - \kappa(\alpha+\beta)c_n = o(1)\|(u_n,v_n)\|_{\mathscr{D}}. \tag{4.7}$$

(1) If $\kappa>0$, for the case of $s_2\le s_1$, we have

$$\begin{aligned} & m + o(1)\big(1+\|(u_n,v_n)\|_{\mathscr{D}}\big) \\ &= \Phi(u_n,v_n) - \frac{1}{2^*(s_1)}J(u_n,v_n) \\ &= \Big(\frac{1}{2}-\frac{1}{2^*(s_1)}\Big)a_n + \Big(\frac{2^*(s_2)}{2^*(s_1)}-1\Big)c_n \\ &\ge \Big(\frac{1}{2}-\frac{1}{2^*(s_1)}\Big)\|(u_n,v_n)\|^2_{\mathscr{D}} \end{aligned}$$

and for the case of $s_2 > s_1$, we have

$$\begin{aligned}
&m + o(1)\big(1 + \|(u_n, v_n)\|_{\mathscr{D}}\big) \\
&= \Phi(u_n, v_n) - \frac{1}{2^*(s_2)} J(u_n, v_n) \\
&= \big(\frac{1}{2} - \frac{1}{2^*(s_2)}\big) a_n + \big(\frac{1}{2^*(s_2)} - \frac{1}{2^*(s_1)}\big) b_n \\
&\geq \big(\frac{1}{2} - \frac{1}{2^*(s_2)}\big) \|(u_n, v_n)\|_{\mathscr{D}}^2.
\end{aligned}$$

(2) If $\kappa < 0, s_2 \geq s_1$, similarly we obtain that

$$m + o(1)\big(1 + \|(u_n, v_n)\|_{\mathscr{D}}\big) \geq \big(\frac{1}{2} - \frac{1}{2^*(s_2)}\big) \|(u_n, v_n)\|_{\mathscr{D}}^2.$$

Based on the above arguments, we can see that $\{(u_n, v_n)\}$ is bounded in $\mathscr{D}$. $\square$

Lemma 4.3. *Under the assumptions of Lemma 4.1, let $\{(u_n, v_n)\} \subset \mathcal{N}$ be a $(PS)_c$ sequence for $\Phi\big|_{\mathcal{N}}$, i.e., $\Phi(u_n, v_n) \to c$ and $\Phi'\big|_{\mathcal{N}}(u_n, v_n) \to 0$ in $\mathscr{D}^*$. Then $\{(u_n, v_n)\}$ is also a $(PS)_c$ sequence for Φ.*

Proof. For any $(u, v) \in \mathcal{N}$, we will follow the previous marks a, b, c defined by (4.2). Then we have

$$a - b - \kappa 2^*(s_2) c = 0$$

and

$$\langle J'(u, v), (u, v) \rangle = 2a - 2^*(s_1) b - \kappa \big(2^*(s_2)\big)^2 c.$$

(1) If $\kappa > 0$,

$$\begin{aligned}
\langle J'(u, v), (u, v) \rangle &= 2\big[b + \kappa 2^*(s_2) c\big] - 2^*(s_1) b - k\big(2^*(s_2)\big)^2 c \\
&= \big[2 - 2^*(s_1)\big] b + \big(2 - 2^*(s_2)\big) 2^*(s_2) \kappa c \\
&< \max\{2 - 2^*(s_1), 2 - 2^*(s_2)\} \big[b + 2^*(s_2) \kappa c\big] \\
&= \max\{2 - 2^*(s_1), 2 - 2^*(s_2)\} a.
\end{aligned}$$

(2) If $\kappa < 0, s_2 \geq s_1$,

$$\begin{aligned}
\langle J'(u, v), (u, v) \rangle &= 2a - 2^*(s_1)\big[a - \kappa(\alpha + \beta) c\big] - \kappa(\alpha + \beta)^2 c \\
&= \big[2 - 2^*(s_1)\big] a + \kappa \big[2^*(s_1) - \alpha - \beta\big](\alpha + \beta) c \\
&\leq \big[2 - 2^*(s_1)\big] a.
\end{aligned}$$

Hence, by (4.5), we obtain that

$$\langle J'(u, v), (u, v) \rangle \leq \max\{2 - 2^*(s_1), 2 - \alpha - \beta\} \delta_0^2 < 0 \text{ for all } (u, v) \in \mathcal{N}, \quad (4.8)$$

where δ_0 is given by (4.5). By the similar arguments as in Lemma 4.2, we can prove that $\{(u_n, v_n)\}$ is bounded in $\mathscr{D}$. Let $\{t_n\} \subset \mathbb{R}$ be a sequence of multipliers satisfying

$$\Phi'(u_n, v_n) = \Phi'|_{\mathcal{N}}(u_n, v_n) + t_n J'(u_n, v_n).$$

Testing by (u_n, v_n), we obtain that

$$t_n \langle J'(u_n, v_n), (u_n, v_n) \rangle \to 0.$$

Recalling (4.8), we obtain $t_n \to 0$. We can also have that $J'(u_n, v_n)$ is bounded due to the boundedness of (u_n, v_n). Hence, it follows that $\Phi'(u_n, v_n) \to 0$ in $\mathscr{D}^*$. $\square$

Define

$$c_0 := \inf_{(u,v)\in\mathcal{N}} \Phi(u, v) \tag{4.9}$$

and

$$\eta := \frac{1}{2} - \frac{1}{2^*(s_{max})},$$

where $s_{max} := \max\{s_1, s_2\}$. From the arguments in the proof of Lemma 4.2, we obtain that

$$c_0 \geq \eta \|(u, v)\|_{\mathscr{D}}^2. \tag{4.10}$$

Combined with Lemma 4.1, we have

$$c_0 \geq \eta \delta_0^2, \tag{4.11}$$

where δ_0 is given by (4.5). If m_0 is achieved by some $(u, v) \in \mathcal{N}$, then (u, v) is a ground state solution of (3.3).

5 Nonexistence of nontrivial ground state solution

In this section, we continue to study the equation (4.1).

Definition 5.1. *In the sequel, we call (u, v) nontrivial iff $u \neq 0$ and $v \neq 0$, and call (u, v) semi-trivial iff either $u = 0$ or $v = 0$ but not all zero.*

We obtain the nonexistence of nontrivial ground state solution of (4.1), i.e., the least energy $c_0 := \inf_{(u,v)\in\mathcal{N}} \Phi(u, v)$ defined in (4.9) can only be attained by semi-trivial pairs. Denote

$$\mu_s(\mathbb{R}_+^N) := \inf\Big\{ \frac{\int_{\mathbb{R}_+^N} |\nabla u|^2 dx}{\big(\int_{\mathbb{R}_+^N} \frac{|u|^{2^*(s)}}{|x|^s} dx\big)^{\frac{2}{2^*(s)}}} : \ u \in D_0^{1,2}(\mathbb{R}_+^N)\backslash\{0\} \Big\}. \tag{5.1}$$

By the result of Egnell [10], $\mu_{s_1}(\mathbb{R}_+^N)$ is achieved and the extremals are parallel to $U(x)$, a ground state solution of the following problem:

$$\begin{cases} -\Delta u = \mu_{s_1}(\mathbb{R}_+^N) \frac{u^{2^*(s_1)-1}}{|x|^{s_1}} \text{ in } \mathbb{R}_+^N, \\ u = 0 \ \text{ on } \partial\mathbb{R}_+^N. \end{cases} \tag{5.2}$$

Define the functional

$$\Psi_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N_+} |\nabla u|^2 dx - \frac{\lambda}{2^*(s_1)}\int_{\mathbb{R}^N_+} \frac{|u|^{2^*(s_1)}}{|x|^{s_1}} dx. \tag{5.3}$$

Then a direct computation shows that u is a least energy critical point of Ψ_λ if and only if

$$u = U_\lambda := \big(\frac{\mu_{s_1}(\mathbb{R}^N_+)}{\lambda}\big)^{\frac{1}{2^*(s_1)-2}} U, \tag{5.4}$$

where U is a ground state solution of (5.2). And the corresponding ground state value is denoted by

$$m_\lambda = \Psi_\lambda(U_\lambda) = [\frac{1}{2} - \frac{1}{2^*(s_1)}]\big(\mu_{s_1}(\mathbb{R}^N_+)\big)^{\frac{2^*(s_1)}{2^*(s_1)-2}} \lambda^{-\frac{2}{2^*(s_1)-2}}. \tag{5.5}$$

Then we see that m_λ is decreasing by λ and

$$c_0 \le \min\{m_\lambda, m_\mu\}, \tag{5.6}$$

where c_0 is defined by (4.9).

Theorem 5.1. *Assume that $\alpha + \beta = 2^*(s_2)$. If one of the following conditions is satisfied:*

(i) $\kappa < 0$ *and* $s_2 \ge s_1$*;*

(ii) $\min\{\alpha, \beta\}\frac{(N-s_1)(2-s_2)}{(N-s_2)(2-s_1)} > 2, s_2 \ge s_1$ *and* $\kappa > 0$ *small enough,*

then we have

$$c_0 = \min\{m_\lambda, m_\mu\}.$$

Moreover, c_0 *is achieved by and only by semitrivial solution*

$$\begin{cases} (U_\lambda, 0) & \text{if } \lambda > \mu, \\ (0, U_\mu) & \text{if } \lambda < \mu, \\ (U_\lambda, 0) \text{ or } (0, U_\lambda) & \text{if } \lambda = \mu, \end{cases}$$

where U_λ, U_μ *are defined by* (5.4).

Remark 5.1. *Theorem 5.1 means that the system (4.1) has only semi-trivial ground state under the hypotheses of the theorem.*

Remark 5.2. *If* $s_1 = s_2 = s \in (0, 2)$, $\min\{\alpha, \beta\} > 2$, *we must have* $N = 3$. *In this case, the assumption that "κ is small enough" can be removed (see Theorem 7.4).*

Proof. Without loss of generality, we only prove the case of $\lambda > \mu$. By (5.6), we see that $c_0 \le m_\lambda$. By (4.11), we also have $c_0 > 0$. Now,we proceed by contradiction. Assume that c_0 is achieved by some $(u, v) \in \mathscr{D}$ such that $u \ne 0, v \ne 0$. Without loss of generality, we may assume that $u \ge 0, v \ge 0$ since c_0 is the least energy.

(i) If $\kappa < 0$, we obtain that

$$\int_{\mathbb{R}^N_+} |\nabla u|^2 - \lambda \int_{\mathbb{R}^N_+} \frac{|u|^{2^*(s_1)}}{|x|^{s_1}} dx = \kappa\alpha \int_{\mathbb{R}^N_+} \frac{|u|^{\alpha}|v|^{\beta}}{|x|^{s_2}} dx \le 0.$$

Recalling that

$$\mu_{s_1}(\mathbb{R}^N_+)\Big(\int_{\mathbb{R}^N_+} \frac{|u|^{2^*(s_1)}}{|x|^{s_1}} dx\Big)^{\frac{2}{2^*(s_1)}} \le \|u\|^2,$$

if $u \neq 0$, we obtain that

$$\int_{\mathbb{R}^N_+} \frac{|u|^{2^*(s_1)}}{|x|^{s_1}} dx \ge \big(\frac{\mu_{s_1}(\mathbb{R}^N_+)}{\lambda}\big)^{\frac{2^*(s_1)}{2^*(s_1)-2}}. \tag{5.7}$$

Then by $s_2 \ge s_1$,

$$\begin{aligned}
&\big(\frac{1}{2} - \frac{1}{2^*(s_2)}\big)\|u\|^2 + \big(\frac{1}{2^*(s_2)} - \frac{1}{2^*(s_1)}\big)\lambda \int_{\mathbb{R}^N_+} \frac{|u|^{2^*(s_1)}}{|x|^{s_1}} dx \\
&\ge \big(\frac{1}{2} - \frac{1}{2^*(s_2)}\big)\mu_{s_1}(\mathbb{R}^N_+)\big(\int_{\mathbb{R}^N_+} \frac{|u|^{2^*(s_1)}}{|x|^{s_1}} dx\big)^{\frac{2}{2^*(s_1)}} + \\
&\quad \big(\frac{1}{2^*(s_2)} - \frac{1}{2^*(s_1)}\big)\lambda \int_{\mathbb{R}^N_+} \frac{|u|^{2^*(s_1)}}{|x|^{s_1}} dx \\
&\ge \big(\frac{1}{2} - \frac{1}{2^*(s_2)}\big)\lambda^{-\frac{2}{2^*(s_1)-2}} \big(\mu_{s_1}(\mathbb{R}^N_+)\big)^{\frac{2^*(s_1)}{2^*(s_1)-2}} + \\
&\quad \big(\frac{1}{2^*(s_2)} - \frac{1}{2^*(s_1)}\big)\lambda^{-\frac{2}{2^*(s_1)-2}} \big(\mu_{s_1}(\mathbb{R}^N_+)\big)^{\frac{2^*(s_1)}{2^*(s_1)-2}} \\
&= \big(\frac{1}{2} - \frac{1}{2^*(s_1)}\big)\lambda^{-\frac{2}{2^*(s_1)-2}} \big(\mu_{s_1}(\mathbb{R}^N_+)\big)^{\frac{2^*(s_1)}{2^*(s_1)-2}} \\
&= m_\lambda.
\end{aligned}$$

Similarly, if $v \neq 0$, we have

$$\int_{\mathbb{R}^N_+} \frac{|v|^{2^*(s_1)}}{|x|^{s_1}} dx \ge \big(\frac{\mu_{s_1}(\mathbb{R}^N_+)}{\mu}\big)^{\frac{2^*(s_1)}{2^*(s_1)-2}} \tag{5.8}$$

and

$$\big(\frac{1}{2} - \frac{1}{2^*(s_2)}\big)\|v\|^2 + \big(\frac{1}{2^*(s_2)} - \frac{1}{2^*(s_1)}\big)\mu \int_{\mathbb{R}^N_+} \frac{|v|^{2^*(s_1)}}{|x|^{s_1}} dx \ge m_\mu > m_\lambda.$$

Then,

$$
\begin{aligned}
c_0 =&\Phi(u,v) = \big(\frac{1}{2}-\frac{1}{2^*(s_2)}\big)a(u,v) + \big(\frac{1}{2^*(s_2)}-\frac{1}{2^*(s_1)}\big)b(u,v)\\
=&\big(\frac{1}{2}-\frac{1}{2^*(s_2)}\big)\|u\|^2 + \big(\frac{1}{2^*(s_2)}-\frac{1}{2^*(s_1)}\big)\lambda\int_{\mathbb{R}^N_+}\frac{|u|^{2^*(s_1)}}{|x|^{s_1}}dx\\
&+\big(\frac{1}{2}-\frac{1}{2^*(s_2)}\big)\|v\|^2 + \big(\frac{1}{2^*(s_2)}-\frac{1}{2^*(s_1)}\big)\mu\int_{\mathbb{R}^N_+}\frac{|v|^{2^*(s_1)}}{|x|^{s_1}}dx\\
\geq&\begin{cases} m_\lambda & \text{if } v=0\\ m_\lambda+m_\mu & \text{if } v\neq 0.\end{cases}
\end{aligned}
$$

Hence, $c_0 = m_\lambda$ is proved and we see that $v=0$, i.e., $(u,v)=(U_\lambda,0)$.

(ii) If $\kappa>0$, we denote

$$
\sigma := \int_{\mathbb{R}^N_+}\frac{|u|^{2^*(s_1)}}{|x|^{s_1}}dx,\ \delta := \int_{\mathbb{R}^N_+}\frac{|v|^{2^*(s_1)}}{|x|^{s_1}}dx.
$$

Then we have

$$
\sigma \leq \big(\frac{\mu_{s_1}(\mathbb{R}^N_+)}{\lambda}\big)^{\frac{2^*(s_1)}{2^*(s_1)-2}}.
$$

If not, apply the above similar arguments, we have

$$
\Phi(u,v) \geq \big(\frac{1}{2}-\frac{1}{2^*(s_2)}\big)\|u\|^2 + \big(\frac{1}{2^*(s_2)}-\frac{1}{2^*(s_1)}\big)\lambda\int_{\mathbb{R}^N_+}\frac{|u|^{2^*(s_1)}}{|x|^{s_1}}dx > m_\lambda,
$$

a contradiction. Similarly, we also have

$$
\delta \leq \big(\frac{\mu_{s_1}(\mathbb{R}^N_+)}{\mu}\big)^{\frac{2^*(s_1)}{2^*(s_1)-2}}.
$$

Similar to the arguments of Lemma 4.1, we have $\Phi(u,v)\geq \big(\frac{1}{2}-\frac{1}{2^*(s_2)}\big)\|(u,v)\|^2_{\mathscr{D}}$. Hence, $\|u\|^2,\|v\|^2\leq \big(\frac{1}{2}-\frac{1}{2^*(s_2)}\big)^{-1}c_0$. By Corollary 2.2, under the assumption of $\min\{\alpha,\beta\}\frac{(N-s_1)(2-s_2)}{(N-s_2)(2-s_1)}>2$, we can choose some proper $\eta_1\geq 2,\eta_2\geq 2$ and $C>0$ such that

$$
\int_{\mathbb{R}^N_+}\frac{|u|^\alpha|v|^\beta}{|x|^{s_2}}dx \leq C|u|^{\eta_1}_{2^*(s_1),s_1} = C\sigma^{\frac{\eta_1}{2^*(s_1)}} \tag{5.9}
$$

and

$$
\int_{\mathbb{R}^N_+}\frac{|u|^\alpha|v|^\beta}{|x|^{s_2}}dx \leq C|v|^{\eta_2}_{2^*(s_1),s_1} = C\delta^{\frac{\eta_2}{2^*(s_1)}}. \tag{5.10}
$$

It follows that there exists some $C>0$ such that

$$
\mu_{s_1}(\mathbb{R}^N_+)\sigma^{\frac{2}{2^*(s_1)}} - \lambda\sigma \leq \kappa C\sigma^{\frac{\eta_1}{2^*(s_1)}} \tag{5.11}
$$

and that

$$
\mu_{s_1}(\mathbb{R}^N_+)\delta^{\frac{2}{2^*(s_1)}} - \mu\delta \leq \kappa C\delta^{\frac{\eta_2}{2^*(s_1)}}. \tag{5.12}
$$

Define $g_i : \mathbb{R}_+ \mapsto \mathbb{R}_+, i = 1, 2$ with

$$g_1(t) := \lambda t^{\frac{2^*(s_1)-2}{2^*(s_1)}} + \kappa C t^{\frac{\eta_1-2}{2^*(s_1)}}$$

and

$$g_2(t) := \mu t^{\frac{2^*(s_1)-2}{2^*(s_1)}} + \kappa C t^{\frac{\eta_2-2}{2^*(s_1)}}.$$

Since $\eta_1, \eta_2 \geq 2$, $g_i(t)$ is strictly increasing in terms of t. It is easy to check that there exists some $\kappa_0 > 0$ such that when $\kappa < \kappa_0$, a direct calculation shows that

$$g_1\Big(\frac{1}{2}\big(\frac{\mu_{s_1}(\mathbb{R}^N_+)}{\lambda}\big)^{\frac{2^*(s_1)}{2^*(s_1)-2}}\Big) < \mu_{s_1}(\mathbb{R}^N_+)$$

and

$$g_2\Big(\frac{1}{2}\big(\frac{\mu_{s_1}(\mathbb{R}^N_+)}{\mu}\big)^{\frac{2^*(s_1)}{2^*(s_1)-2}}\Big) < \mu_{s_1}(\mathbb{R}^N_+).$$

Hence, if $\sigma \neq 0, \delta \neq 0$, by (5.11) and (5.12), we obtain that

$$\sigma > \frac{1}{2}\big(\frac{\mu_{s_1}(\mathbb{R}^N_+)}{\lambda}\big)^{\frac{2^*(s_1)}{2^*(s_1)-2}}$$

and

$$\delta > \frac{1}{2}\big(\frac{\mu_{s_1}(\mathbb{R}^N_+)}{\mu}\big)^{\frac{2^*(s_1)}{2^*(s_1)-2}}.$$

Then

$$\begin{aligned}
&\big(\frac{1}{2} - \frac{1}{2^*(s_2)}\big)\|u\|^2 + \big(\frac{1}{2^*(s_2)} - \frac{1}{2^*(s_1)}\big)\lambda \int_{\mathbb{R}^N_+} \frac{|u|^{2^*(s_1)}}{|x|^{s_1}} dx \\
\geq &\big(\frac{1}{2} - \frac{1}{2^*(s_2)}\big)\mu_{s_1}(\mathbb{R}^N_+)\sigma^{\frac{2}{2^*(s_1)}} + \big(\frac{1}{2^*(s_2)} - \frac{1}{2^*(s_1)}\big)\lambda\sigma \\
\geq &\big(\frac{1}{2} - \frac{1}{2^*(s_2)}\big)\mu_{s_1}(\mathbb{R}^N_+)\big[\frac{1}{2}\big(\frac{\mu_{s_1}(\mathbb{R}^N_+)}{\lambda}\big)^{\frac{2^*(s_1)}{2^*(s_1)-2}}\big]^{\frac{2}{2^*(s_1)}} \\
&+ \big(\frac{1}{2^*(s_2)} - \frac{1}{2^*(s_1)}\big)\lambda\frac{1}{2}\big(\frac{\mu_{s_1}(\mathbb{R}^N_+)}{\lambda}\big)^{\frac{2^*(s_1)}{2^*(s_1)-2}} \\
> &\frac{1}{2}\big(\frac{1}{2} - \frac{1}{2^*(s_1)}\big)\lambda^{-\frac{2}{2^*(s_1)-2}}\mu_{s_1}(\mathbb{R}^N_+)^{\frac{2^*(s_1)}{2^*(s_1)-2}} \\
= &\frac{1}{2}m_\lambda.
\end{aligned}$$

Similarly, we have

$$\big(\frac{1}{2} - \frac{1}{2^*(s_2)}\big)\|v\|^2 + \big(\frac{1}{2^*(s_2)} - \frac{1}{2^*(s_1)}\big)\mu \int_{\mathbb{R}^N_+} \frac{|v|^{2^*(s_1)}}{|x|^{s_1}} dx > \frac{1}{2}m_\mu > \frac{1}{2}m_\lambda.$$

Thus,

$$
\begin{aligned}
\Phi(u,v) =&\big(\frac{1}{2}-\frac{1}{2^*(s_2)}\big)\|u\|^2+\big(\frac{1}{2^*(s_2)}-\frac{1}{2^*(s_1)}\big)\lambda\int_{\mathbb{R}^N_+}\frac{|u|^{2^*(s_1)}}{|x|^{s_1}}dx\\
&+\big(\frac{1}{2}-\frac{1}{2^*(s_2)}\big)\|v\|^2+\big(\frac{1}{2^*(s_2)}-\frac{1}{2^*(s_1)}\big)\mu\int_{\mathbb{R}^N_+}\frac{|v|^{2^*(s_1)}}{|x|^{s_1}}dx\\
>&m_\lambda,
\end{aligned}
$$

a contradiction.

The arguments above imply that $\sigma = 0$ or $\delta = 0$, i.e., $u = 0$ or $v = 0$. If $u = 0$, then $v \neq 0$ is a critical point of Ψ_μ and then $\Phi(u,v) = \Psi_\mu(v) \geq m_\mu > m_\lambda$, also a contradiction. Thus, we obtain that $u \neq 0, v = 0$. Hence u is a critical point of Ψ_λ, and $c_0 = \Phi(u,v) = \Psi_\lambda(u) \geq m_\lambda$. Then, we have $c_0 = m_\lambda$ and $u = U_\lambda$. $\square$

Remark 5.3. *We remark that the Theorem 5.1 of this section is valid for any cone Ω.*

6 Preliminaries for the existence results

Remark 6.1. *Without loss of generality, we only consider the case of $\Omega = \mathbb{R}^N_+$. We remark that the results of this section are still valid for any domain Ω as long as $\mu_{s_1}(\Omega)$ is attained (e.g. Ω is a cone).*

Since the system (4.1) possesses semitrivial solution (u,v), we are interested in the nontrivial solutions. Firstly, we recall the following result due to Ghoussoub and Robert [13, Theorem 1.2] (see also [15, Lemma 2.1], [19, Lemma 2.6]) for the scalar equation.

Lemma 6.1. *([13, Theorem 1.2]) Let $u \in D^{1,2}_0(\mathbb{R}^N_+)$ be an entire solution to the problem*

$$
\begin{cases}
\Delta u + \frac{u^{2^*(s_1)-1}}{|x|^{s_1}} = 0 & \text{in } \mathbb{R}^N_+,\\
u > 0 \ \text{ in } \mathbb{R}^N_+ \ \text{ and} & u = 0 \text{ on } \partial\mathbb{R}^N_+.
\end{cases}
\tag{6.1}
$$

Then, the following hold:

(i)

$$
\begin{cases}
u \in C^2(\overline{\mathbb{R}^N_+}) & \text{if } s_1 < 1 + \frac{2}{N},\\
u \in C^{1,\beta}(\overline{\mathbb{R}^N_+}) & \text{for all } 0 < \beta < 1 \text{ if } s_1 = 1 + \frac{2}{N},\\
u \in C^{1,\beta}(\overline{\mathbb{R}^N_+}) & \text{for all } 0 < \beta < \frac{N(2-s_1)}{N-2} \text{ if } s_1 > 1 + \frac{2}{N}.
\end{cases}
$$

(ii) There is a constant C such that $|u(x)| \leq C(1+|x|)^{1-N}$ and $|\nabla u(x)| \leq C(1+|x|)^{-N}$.

(iii) $u(x', x_N)$ is axially symmetric with respect to the x_N-axis, i.e., $u(x', x_N) = u(|x'|, x_N)$, where $x' = (x_1, \cdots, x_{N-1})$.

6.1 Existence of positive solution for the case : $\lambda = \mu(\frac{\beta}{\alpha})^{\frac{2^*(s_1)-2}{2}}$

The following result is essentially due to [17, Theorem 1.2]:

Lemma 6.2. *Let $N \geq 3$, $s_1, s_2 \in (0,2), \lambda > 0$, $\alpha > 1, \beta > 1$ and $\alpha + \beta = 2^*(s_2)$. Then the following problem*

$$\begin{cases} -\Delta w - \lambda \frac{w^{2^*(s_1)-1}}{|x|^{s_1}} - \kappa\alpha(\frac{\beta}{\alpha})^{\frac{\beta}{2}} \frac{w^{2^*(s_2)-1}}{|x|^{s_2}} = 0 \text{ in } \mathbb{R}_+^N, \\ w(x) \in D_0^{1,2}(\mathbb{R}_+^N),\ w(x) > 0 \text{ in } \mathbb{R}_+^N, \end{cases} \tag{6.2}$$

has a least-energy solution provided further one of the following holds:

(i) $0 < s_1 < s_2 < 2$ *and* $\kappa \in \mathbb{R}$.

(ii) $s_1 > s_2$ *and* $\kappa \geq 0$.

(iii) $s_1 = s_2$ *and* $\kappa > -\lambda\frac{1}{\alpha}(\frac{\alpha}{\beta})^{\frac{\beta}{2}}$.

Remark 6.2. *The case of $\kappa = 0$ or $s_1 = s_2$ with $\kappa > -\lambda\frac{1}{\alpha}(\frac{\alpha}{\beta})^{\frac{\beta}{2}}$,* (6.2) *is essentially the problem* (6.1). *And the existence result was firstly given by Egnell [10].*

Corollary 6.1. *Let $N \geq 3$, $s_1, s_2 \in (0,2), \mu > 0, \kappa \neq 0$, $\alpha > 1, \beta > 1$ and $\alpha + \beta = 2^*(s_2)$. If $\lambda = \mu(\frac{\beta}{\alpha})^{\frac{2^*(s_1)}{2}}$, then $(w, \sqrt{\frac{\beta}{\alpha}}w)$ is a positive solution of* (3.3) *provided further one of the following holds:*

(i) $0 < s_1 < s_2 < 2$ *and* $\kappa \in \mathbb{R}\backslash\{0\}$.

(ii) $s_1 > s_2$ *and* $\kappa > 0$.

(iii) $s_1 = s_2$ *and* $\kappa > -\lambda\frac{1}{\alpha}(\frac{\alpha}{\beta})^{\frac{\beta}{2}}$.

Here w is a least-energy solution of (6.2).

Proof. This proof can be got through via a direct computation. We omit the details. ☐

Corollary 6.2. *Assume that $N \geq 3, \alpha > 1, \beta > 1, \alpha+\beta = 2^*(s_2), \lambda = \mu(\frac{\beta}{\alpha})^{\frac{2^*(s_1)}{2}} > 0$ and one of the following holds*

(i) $0 < s_1 < s_2 < 2, \kappa < 0$,

(ii) $s_1 = s_2 \in (0,2), -\lambda\frac{1}{\alpha}(\frac{\alpha}{\beta})^{\frac{\beta}{2}} < \kappa < 0$,

(iii) $\min\{\alpha,\beta\}\frac{(N-s_1)(2-s_2)}{(N-s_2)(2-s_1)} > 2, s_2 \geq s_1$ *and* $\kappa > 0$ *small enough.*

Then $(w, \sqrt{\frac{\beta}{\alpha}}w)$ is a positive solution to equation (3.3) *but problem* (3.3) *has no nontrivial ground state solution.*

Proof. It is a straightforward consequence of Theorem 5.1 and Corollary 6.1. ☐

6.2 Estimation on the upper bound of $c_0 := \inf\limits_{(u,v)\in\mathcal{N}} \Phi(u,v)$

In order to prove the existence of positive ground state solution to the equation (4.1), we have to give an estimation on the upper bound of c_0, including the cases of $s_1 = s_2$ and $s_1 \neq s_2$.

Let $1 < \alpha, 1 < \beta, \alpha + \beta = 2^*(s_2)$. Let $u := U_\lambda$ be a function defined by (5.4). Then we have $u > 0$ in $\mathbb{R}^N_+$ and

$$c(u,v) := \int_{\mathbb{R}^N_+} \frac{|u|^\alpha |v|^\beta}{|x|^{s_2}} dx > 0, \quad \forall\ v \in D_0^{1,2}(\mathbb{R}^N_+)\backslash\{0\}.$$

Assume that the assumptions of Lemma 4.1 are satisfied, i.e., one of the following holds:

(i) $\kappa > 0$,

(ii) $\kappa < 0$ and $s_2 > s_1$,

(iii) $s_2 = s_1$ and $\kappa < 0$ small enough,

then we see that for any $\varepsilon \in \mathbb{R}$, there exists a unique positive number $t(\varepsilon) > 0$ such that $\big(t(\varepsilon)u, t(\varepsilon)\varepsilon v\big) \in \mathcal{N}$. The function $t(\varepsilon) : \mathbb{R} \mapsto \mathbb{R}_+$ is implicitly defined by the equation

$$\begin{aligned}\|u\|^2 + \varepsilon^2\|v\|^2 =& \Big[\lambda|u|^{2^*(s_1)}_{2^*(s_1),s_1} + \mu|v|^{2^*(s_1)}_{2^*(s_1),s_1}|\varepsilon|^{2^*(s_1)}\Big]\big[t(\varepsilon)\big]^{2^*(s_1)-2}\\ &+ \kappa 2^*(s_2)c(u,v)\big[t(\varepsilon)\big]^{2^*(s_2)-2}|\varepsilon|^\beta.\end{aligned} \tag{6.3}$$

We notice that $t(0) = 1$. Moreover, from the Implicit Function Theorem, it follows that $t(\varepsilon) \in C^1(\mathbb{R})$ and $t'(\varepsilon) = \frac{P_v(\varepsilon)}{Q_v(\varepsilon)}$, where

$$\begin{aligned}Q_v(\varepsilon) :=& \big[2^*(s_1) - 2\big]\Big[\lambda|u|^{2^*(s_1)}_{2^*(s_1),s_1} + \mu|v|^{2^*(s_1)}_{2^*(s_1),s_1}|s|^{2^*(s_1)}\Big]\big[t(\varepsilon)\big]^{2^*(s_1)-3}\\ &+ \kappa 2^*(s_2)[2^*(s_2)-2]c(u,v)\big[t(\varepsilon)\big]^{2^*(s_2)-3}|\varepsilon|^\beta\end{aligned}$$

and

$$\begin{aligned}P_v(\varepsilon) :=& 2\|v\|^2\varepsilon - 2^*(s_1)\mu|v|^{2^*(s_1)}_{2^*(s_1),s_1}\big[t(\varepsilon)\big]^{2^*(s_1)-2}|\varepsilon|^{2^*(s_1)-2}\varepsilon\\ &- \kappa 2^*(s_2)\beta c(u,v)\big[t(\varepsilon)\big]^{2^*(s_2)-2}|\varepsilon|^{\beta-2}\varepsilon.\end{aligned}$$

Lemma 6.3. *(***The case of** $\beta < 2$*) Assume that* $1 < \alpha, 1 < \beta < 2, \alpha + \beta = 2^*(s_2)$ *and one of the following holds:*

(i) $\kappa > 0$.

(ii) $\kappa < 0$ *and* $s_2 > s_1$.

(iii) $s_2 = s_1$ *and* $\kappa < 0$ *with* $|\kappa|$ *small enough.*

Let $U_\lambda := \big(\frac{\mu_{s_1}(\mathbb{R}^N_+)}{\lambda}\big)^{\frac{1}{2^*(s_1)-2}} U$, where U is a ground state solution of (5.2). *Then*

(a) *if $\kappa < 0$, $(u, 0)$ is a local minimum point of Φ in $\mathcal{N}$.*

(b) *if $\kappa > 0$, then*

$$c_0 := \inf_{(\phi,\varphi)\in\mathcal{N}} \Phi(\phi,\varphi) < \Phi(U_\lambda, 0) = \Psi_\lambda(U_\lambda) = m_\lambda.$$

Proof. Let $u := U_\lambda$ and take any $v \in D_0^{1,2}(\mathbb{R}^N_+)\backslash\{0\}$. When $\beta < 2$, we have

$$P_v(\varepsilon) = -\kappa 2^*(s_2)\beta c(u,v)|\varepsilon|^{\beta-2}\varepsilon\big(1 + o(1)\big) \text{ as } \varepsilon \to 0$$

and

$$Q_v(\varepsilon) = \Big(\big[2^*(s_1) - 2\big]\lambda |u|^{2^*(s_1)}_{2^*(s_1),s_1}\Big)\big(1 + o(1)\big) \text{ as } \varepsilon \to 0.$$

Hence,

$$t'(\varepsilon) = -M(v)\beta|\varepsilon|^{\beta-2}\varepsilon\big(1 + o(1)\big),$$

where

$$M(v) := \frac{\kappa 2^*(s_2)c(u,v)}{\big[2^*(s_1) - 2\big]\lambda|u|^{2^*(s_1)}_{2^*(s_1),s_1}}.$$

By the Taylor formula, we obtain that

$$t(\varepsilon) = 1 - M(v)|\varepsilon|^\beta\big(1 + o(1)\big),$$

$$\big[t(\varepsilon)\big]^{2^*(s_1)} = 1 - 2^*(s_1)M(v)|\varepsilon|^\beta\big(1 + o(1)\big),$$

and

$$\big[t(\varepsilon)\big]^{2^*(s_2)} = 1 - 2^*(s_2)M(v)|\varepsilon|^\beta\big(1 + o(1)\big).$$

Noting that for any $(\phi,\varphi) \in \mathcal{N}$, we have

$$\Phi(\phi,\varphi) = (\frac{1}{2} - \frac{1}{2^*(s_1)})b(\phi,\varphi) + \frac{2^*(s_2) - 2}{2}\kappa c(\phi,\varphi),$$

where $b(\phi,\varphi), c(\phi,\varphi)$ are defined by (4.2). Thus,

$$\begin{aligned}
&\Phi\big(t(\varepsilon)u, t(\varepsilon)\varepsilon v\big) - \Phi(u,0)\\
=&(\frac{1}{2} - \frac{1}{2^*(s_1)})\Big[\lambda|u|^{2^*(s_1)}_{2^*(s_1),s_1}\big[t(\varepsilon)\big]^{2^*(s_1)} + \mu|v|^{2^*(s_1)}_{2^*(s_1),s_1}\big[t(\varepsilon)\big]^{2^*(s_1)}|\varepsilon|^{2^*(s_1)}\\
&- \lambda|u|^{2^*(s_1)}_{2^*(s_1),s_1}\Big] + \frac{2^*(s_2) - 2}{2}\kappa c(u,v)\big[t(\varepsilon)\big]^{2^*(s_2)}|\varepsilon|^\beta\\
=&(\frac{1}{2} - \frac{1}{2^*(s_1)})\lambda|u|^{2^*(s_1)}_{2^*(s_1),s_1}\big[-2^*(s_1)M(v)\big]|\varepsilon|^\beta\big(1 + o(1)\big)\\
&+ \frac{2^*(s_2) - 2}{2}\kappa c(u,v)\big[t(\varepsilon)\big]^{2^*(s_2)}|\varepsilon|^\beta\\
=& - \frac{\kappa 2^*(s_2)}{2}c(u,v)|\varepsilon|^\beta\big(1 + o(1)\big) + \frac{2^*(s_2) - 2}{2}\kappa c(u,v)\big(1 + o(1)\big)|\varepsilon|^\beta\\
=& - \kappa|\varepsilon|^\beta c(u,v)\big(1 + o(1)\big),
\end{aligned}$$

which implies the results since $c(u,v) > 0$ for any $0 \neq v \in D_0^{1,2}(\mathbb{R}^N_+)$. $\square$

Lemma 6.4. *(***the case of** $\beta > 2$*) Assume that* $1 < \alpha, 2 < \beta, \alpha + \beta = 2^*(s_2)$ *and one of the following holds:*

(i) $\kappa > 0$.

(ii) $\kappa < 0$ *and* $s_2 > s_1$.

(iii) $s_2 = s_1$ *and* $\kappa < 0$ *with* $|\kappa|$ *small enough.*

Let $U_\lambda := \big(\frac{\mu_{s_1}(\mathbb{R}^N_+)}{\lambda}\big)^{\frac{1}{2^*(s_1)-2}} U$*, where* U *is a ground state solution of* (5.2)*. Then* $(U_\lambda, 0)$ *is a local minimum point of* Φ *in* $\mathcal{N}$*.*

Proof. Let $u := U_\lambda$ and take any $v \in D_0^{1,2}(\mathbb{R}^N_+)\backslash\{0\}$. When $\beta > 2$, we have

$$P_v(\varepsilon) = 2\|v\|^2\varepsilon\big(1 + o(1)\big)$$

and

$$Q_v(\varepsilon) = \Big(\big[2^*(s_1) - 2\big]\lambda|u|^{2^*(s_1)}_{2^*(s_1),s_1}\Big)\big(1 + o(1)\big).$$

Hence,

$$t'(\varepsilon) = 2\tilde{M}(v)\varepsilon\big(1 + o(1)\big),$$

where

$$\tilde{M}(v) := \frac{\|v\|^2}{\big[2^*(s_1) - 2\big]\lambda|u|^{2^*(s_1)}_{2^*(s_1),s_1}}.$$

By the Taylor formula, we obtain that

$$t(\varepsilon) = 1 + \tilde{M}(v)|\varepsilon|^2\big(1 + o(1)\big),$$

$$\big[t(\varepsilon)\big]^{2^*(s_1)} = 1 + 2^*(s_1)\tilde{M}(v)|\varepsilon|^2\big(1 + o(1)\big),$$

and

$$\big[t(\varepsilon)\big]^{2^*(s_2)} = 1 + 2^*(s_2)\tilde{M}(v)|\varepsilon|^2\big(1 + o(1)\big).$$

Hence a direct computation shows that

$$\begin{aligned}
&\Phi\big(t(\varepsilon)u, t(\varepsilon)\varepsilon v\big) - \Phi(u, 0)\\
=&\big(\frac{1}{2} - \frac{1}{2^*(s_1)}\big)\lambda|u|^{2^*(s_1)}_{2^*(s_1),s_1}\big[2^*(s_1)\tilde{M}(v)\big]|\varepsilon|^2\big(1 + o(1)\big) + o(|\varepsilon|^2)\\
=&\frac{1}{2}\|v\|^2|\varepsilon|^2\big(1 + o(1)\big)\\
>&0 \text{ when } \varepsilon \text{ is small enough.}
\end{aligned}$$

$\square$

Define $\eta_1 := \inf\limits_{v\in\Xi} \|v\|^2$, where

$$\Xi := \{v \in D_0^{1,2}(\mathbb{R}^N_+) : \quad \int_{\mathbb{R}^N_+} \frac{|U_\lambda|^\alpha|v|^2}{|x|^{s_2}}dx = 1\}.$$

We note that by the Hardy-Sobolev inequality, $U_\lambda \in L^{2^*(s_2)}(\mathbb{R}^N_+, \frac{dx}{|x|^{s_2}})$, then by the Hölder inequality, $\int_{\mathbb{R}^N_+} \frac{|U_\lambda|^\alpha |v|^2}{|x|^{s_2}} dx$ is well defined for all $v \in D_0^{1,2}(\mathbb{R}^N_+)$ when $\alpha = 2^*(s_2) - 2$.

Define $\langle \phi, \psi \rangle := \int_\Omega \frac{|U_\lambda|^\alpha \phi\psi}{|x|^{s_2}} dx$, then it is easy to check that $\langle \cdot, \cdot \rangle$ is an inner product. We say that ϕ and ψ are orthogonal if and only if $\langle \phi, \psi \rangle = 0$. Then we have the following result:

Lemma 6.5. *Assume that* $1 < \alpha = 2^*(s_2) - 2, \beta = 2$, *then there exists* $\eta_1 > 0$ *and some* $0 < v \in D_0^{1,2}(\mathbb{R}^N_+)$ *such that*

$$\begin{cases} -\Delta v = \eta_1 \frac{|U_\lambda|^\alpha}{|x|^{s_2}} v & \textit{in } \mathbb{R}^N_+ \\ v = 0 & \textit{on } \partial\mathbb{R}^N_+. \end{cases} \tag{6.4}$$

Furthermore, the eigenvalue η_1 *is simple and satisfying*

$$\int_{\mathbb{R}^N_+} \frac{|U_\lambda|^\alpha |v|^2}{|x|^{s_2}} dx \leq \frac{1}{\eta_1} \|v\|^2 \textit{ for all } v \in D_0^{1,2}(\mathbb{R}^N_+). \tag{6.5}$$

In particular, if $s_1 = s_2 = s$, *we have*

$$\eta_1 = \lambda, \tag{6.6}$$

and it is only attained by $v = U_\lambda$.

Proof. It is easy to see that $\eta_1 \geq 0$ under our assumptions. Let $\{v_n\} \subset \Xi$ be such that $\|v_n\|^2 \to \eta_1$. Then $\{v_n\}$ is bounded in $D_0^{1,2}(\mathbb{R}^N_+)$. Going to a subsequence if necessary, we may assume that $v_n \rightharpoonup v_0$ in $D_0^{1,2}(\mathbb{R}^N_+)$ and $v_n \to v_0$ a.e. in $\mathbb{R}^N_+$. By Hölder inequality, we have

$$\Big| \int_\Lambda \frac{u^\alpha v_n^2 - u^\alpha v_0^2}{|x|^{s_2}} dx \Big| \leq \Big(\int_\Lambda \frac{u^{\alpha+2}}{|x|^{s_2}} dx \Big)^{\frac{\alpha}{\alpha+2}} \Big(\int_\Lambda \frac{|v_n^2 - v^2|^{\frac{\alpha+2}{2}}}{|x|^{s_2}} \Big)^{\frac{2}{\alpha+2}} \to 0$$

as $|\Lambda| \to 0$ due to the absolute continuity of the integral and the boundness of v_n. Similarly, we also have that $\{\frac{u^\alpha v_n^2}{|x|^{s_2}}\}$ is a tight sequence, i.e., for $\varepsilon > 0$, there exists some $R_\varepsilon > 0$ such that

$$\Big| \int_{\mathbb{R}^N_+ \cap B^c_{R_\varepsilon}} \frac{u^\alpha v_n^2}{|x|^{s_2}} dx \Big| \leq \varepsilon \text{ uniformly for all } n \in \mathbb{N}. \tag{6.7}$$

Combine with the Egoroff Theorem, it is easy to prove that

$$\int_{\mathbb{R}^N_+} \frac{u^\alpha v_n^2}{|x|^{s_2}} dx \to \int_{\mathbb{R}^N_+} \frac{u^\alpha v_0^2}{|x|^{s_2}} dx. \tag{6.8}$$

Hence, we prove that

$$D_0^{1,2}(\mathbb{R}^N_+) \mapsto \mathbb{R} \text{ with } \chi(v) = \int_{\mathbb{R}^N_+} \frac{|u|^\alpha |v|^2}{|x|^{s_2}} dx$$

is weak continuous, which implies that Ξ is weak closed. Hence, $v_0 \in \Xi$ and we have

$$\|v_0\|^2 \leq \liminf_{n\to\infty} \|v_n\|^2 = \eta_1.$$

On the other hand, by the definition of η_1 and $v_0 \in \Xi$, we have

$$\|v_0\|^2 \geq \eta_1.$$

Thus, v_0 is a minimizer of $\|v\|^2$ constraint on Ξ. It is easy to see that $|v_0|$ is also a minimizer. Hence, we may assume that $v_0 \geq 0$ without loss of generality. We see that there exists some Lagrange multiplier $\eta \in \mathbb{R}$ such that

$$-\Delta v_0 = \eta \frac{|u|^\alpha v_0}{|x|^{s_2}}.$$

It follows that $\eta = \eta_1$. Since $v_0 \in \Xi$, we get that $v_0 \neq 0$ and $\eta_1 > 0$.

Let $a(x) := \eta_1 \frac{|u|^\alpha}{|x|^{s_2}}$, it is easy to see that $a(x) \in L^{\frac{N}{2}}_{loc}(\mathbb{R}^N_+)$. Then the Brézis-Kato theorem in [4] implies that $v \in L^r_{loc}(\mathbb{R}^N_+)$ for all $1 \leq r < \infty$. Then $v_0 \in W^{2,r}_{loc}(\mathbb{R}^N_+)$ for all $1 \leq r < \infty$. By the elliptic regularity theory, $v_0 \in C^2(\mathbb{R}^N_+)$. Finally, by the maximum principle, we obtain that v_0 is positive. Finally, (6.5) is an easy conclusion from the definition of η_1.

It is standard to prove that η_1 is simple, we omit the details. Next, we will compute the value of η_1 when $s_1 = s_2 = s$. A direct computation shows that

$$-\Delta U_\lambda = \lambda \frac{U_\lambda^{2^*(s)-1}}{|x|^s}. \tag{6.9}$$

Testing (6.4) by U_λ, we have

$$\int_{\mathbb{R}^N_+} (\nabla v \cdot \nabla U_\lambda) dx = \eta_1 \int_{\mathbb{R}^N_+} \frac{U_\lambda^\alpha}{|x|^s} v U_\lambda dx. \tag{6.10}$$

Testing (6.9) by v, we also have

$$\int_{\mathbb{R}^N_+} (\nabla U_\lambda \cdot \nabla v) dx = \lambda \int_{\mathbb{R}^N_+} \frac{U_\lambda^\alpha}{|x|^s} U_\lambda v dx. \tag{6.11}$$

Hence,

$$(\eta_1 - \lambda) \int_{\mathbb{R}^N_+} \frac{U_\lambda^\alpha}{|x|^s} U_\lambda v dx = 0. \tag{6.12}$$

Since v and U_λ are positive, we obtain that $\eta_1 = \lambda$. $\square$

Lemma 6.6. *(**the case of** $\beta = 2$) Assume that $1 < \alpha = 2^*(s_2) - 2, \beta = 2$ and one of the following holds:*

(i) $\kappa > 0$.

(ii) $\kappa < 0$ *and* $s_2 > s_1$.

(iii) $s_2 = s_1$ *and* $\kappa < 0$ *with* $|\kappa|$ *small enough.*

Let $U_\lambda := \big(\frac{\mu_{s_1}(\mathbb{R}^N_+)}{\lambda}\big)^{\frac{1}{2^*(s_1)-2}} U$*, where* U *is a ground state solution of* (5.2)*. Then there exists a positive* $k_0 = \frac{\eta_1}{2^*(s_2)}$ *such that*

(a) *if* $\kappa < 0$*,* $(U_\lambda, 0)$ *is a local minimum point of* Φ *in* $\mathcal{N}$*.*

(b) *if* $0 < \kappa < k_0$*, then* $(U_\lambda, 0)$ *is a local minimum point of* Φ *in* $\mathcal{N}$*.*

(c) *if* $\kappa > k_0$*, then*

$$c_0 := \inf_{(\phi,\varphi)\in\mathcal{N}} \Phi(\phi,\varphi) < \Phi(U_\lambda, 0) = \Psi_\lambda(U_\lambda) = m_\lambda,$$

where $\eta_1 := \inf\limits_{v\in\Xi} \|v\|^2$*, with*

$$\Xi := \{v \in D_0^{1,2}(\mathbb{R}^N_+) : \int_{\mathbb{R}^N_+} \frac{|U_\lambda|^\alpha |v|^2}{|x|^{s_2}} dx = 1\}.$$

Proof. Let $u := U_\lambda$ and take any $v \in D_0^{1,2}(\mathbb{R}^N_+)\backslash\{0\}$. In this case, we have

$$P_v(\varepsilon) = 2\Big(\|v\|^2 - \kappa 2^*(s_2) c(u,v)\Big)\varepsilon\big(1 + o(1)\big)$$

and

$$Q_v(\varepsilon) = \Big(\big[2^*(s_1) - 2\big]\lambda |u|^{2^*(s_1)}_{2^*(s_1),s_1}\Big)\big(1 + o(1)\big).$$

Hence,

$$t'(\varepsilon) = 2\bar{M}(v)\varepsilon\big(1 + o(1)\big),$$

where

$$\bar{M}(v) := \frac{\|v\|^2 - \kappa 2^*(s_2) c(u,v)}{\big[2^*(s_1) - 2\big]\lambda |u|^{2^*(s_1)}_{2^*(s_1),s_1}}.$$

Similar to the arguments above, we obtain that

$$\Phi\big(t(s)u, t(s)sv\big) - \Phi(u, 0) = \frac{1}{2}\big(\|v\|^2 - \kappa 2^*(s_2) c(u,v)\big)|s|^2\big(1 + o(1)\big).$$

If $\kappa < 0$, we obtain the result of (a).

Since $u > 0$ is given, by Lemma 6.5, we have

$$c(u,v) = \int_{\mathbb{R}^N_+} \frac{|u|^\alpha |v|^2}{|x|^{s_2}} dx \le \frac{1}{\eta_1}\|v\|^2 \text{ for all } v \in D_0^{1,2}(\mathbb{R}^N_+). \tag{6.13}$$

Define

$$k_0 := \frac{\eta_1}{2^*(s_2)}.$$

Hence, when $\kappa < k_0$, we have

$$\|v\|^2 - 2^*(s_2)\kappa c(u,v) > 0 \text{ for all } v \in D_0^{1,2}(\mathbb{R}^N_+)\backslash\{0\},$$

which implies (b). For $\kappa > k_0 = \frac{\eta_1}{2^*(s_2)}$, by the definition of η_1, there exists some $v \in D_0^{1,2}(\mathbb{R}^N_+)\backslash\{0\}$ such that

$$\|v\|^2 - \kappa 2^*(s_2)c(u,v) < 0.$$

and then it follows (c). □

Summarize the above conclusions, we obtain the following result:

Corollary 6.3. *Assume that* $1 < \alpha, 1 < \beta, \alpha + \beta \leq 2^*(s_2)$ *and one of the following holds:*

(i) $\kappa > 0$.

(ii) $\kappa < 0$ *and* $s_2 > s_1$.

(iii) $s_2 = s_1$ *and* $\kappa < 0$ *with* $|\kappa|$ *small enough.*

Let $U_\lambda := \big(\frac{\mu_{s_1}(\mathbb{R}^N_+)}{\lambda}\big)^{\frac{1}{2^*(s_1)-2}} U$*, where* U *is a ground state solution of* (5.2)*.*

(1) Assume that either $\kappa < 0$ *or* $\beta > 2$ *or* $\beta = 2$ *but with* $\kappa < \frac{\eta_1}{2^*(s_2)}$*, then* $(U_\lambda, 0)$ *is a local minimum point of* Φ *in* $\mathcal{N}$*.*

(2) Assume that either $\beta < 2$ *and* $\kappa > 0$ *or* $\beta = 2$ *but with* $\kappa > \frac{\eta_1}{2^*(s_2)}$*, then*

$$c_0 := \inf_{(\phi,\varphi)\in\mathcal{N}} \Phi(\phi,\varphi) < \Phi(U_\lambda, 0) = \Psi_\lambda(U_\lambda) = m_\lambda,$$

where η_1 *is defined as that in Lemma 6.5. In particular,* $\eta_1 = \lambda$ *if* $s_1 = s_2 = s$*.*

Apply the similar arguments, we can obtain the following result:

Corollary 6.4. *Assume that* $1 < \alpha, 1 < \beta, \alpha + \beta \leq 2^*(s_2)$ *and one of the following holds:*

(i) $\kappa > 0$.

(ii) $\kappa < 0$ *and* $s_2 > s_1$.

(iii) $s_2 = s_1$ *and* $\kappa < 0$ *with* $|\kappa|$ *small enough.*

Let $U_\mu := \big(\frac{\mu_{s_1}(\mathbb{R}^N_+)}{\mu}\big)^{\frac{1}{2^*(s_1)-2}} U$*, where* U *is a ground state solution of* (5.2)*. Then*

(1) if either $\kappa < 0$ *or* $\alpha > 2$ *or* $\alpha = 2$ *but with* $\kappa < \frac{\eta_2}{2^*(s_2)}$*, then* $(0, U_\mu)$ *is a local minimum point of* Φ *in* $\mathcal{N}$*;*

(2) if $\alpha < 2$ and $\kappa > 0$ or $\alpha = 2$ but with $\kappa > \frac{\eta_2}{2^*(s_2)}$, then

$$c_0 = \inf_{(\phi,\varphi)\in\mathcal{N}} \Phi(\phi,\varphi) < \Phi(0,U_\mu) = \Psi_\mu(U_\mu) = m_\mu,$$

where $\eta_2 := \inf_{u\in\Theta} \|u\|^2$, and $\Theta := \Big\{u \in D_0^{1,2}(\mathbb{R}_+^N) : \int_{\mathbb{R}_+^N} \frac{|u|^2|U_\mu|^\beta}{|x|^{s_2}}dx = 1\Big\}$. In particular $\eta_2 = \mu$ whenever $s_1 = s_2 = s$.

7 The case of $s_1 = s_2 = s \in (0,2)$: nontrivial ground state and uniqueness; sharp constant $S_{\alpha,\beta,\lambda,\mu}(\Omega)$; existence of infinitely many sign-changing solutions

In this section, we focus on the case of $s_1 = s_2 := s \in (0,2)$; the case $s_1 \neq s_2$ will be studied in the forthcoming paper (Part II). That is, we study the following problem

$$\begin{cases} -\Delta u - \lambda\frac{|u|^{2^*(s)-2}u}{|x|^s} = \kappa\alpha\frac{1}{|x|^s}|u|^{\alpha-2}u|v|^\beta & \text{in } \Omega,\\ -\Delta v - \mu\frac{|v|^{2^*(s)-2}v}{|x|^s} = \kappa\beta\frac{1}{|x|^s}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega,\\ (u,v) \in \mathscr{D} := D_0^{1,2}(\Omega)\times D_0^{1,2}(\Omega), \end{cases} \tag{7.1}$$

where Ω is a cone in $\mathbb{R}^N$(especially, $\Omega = \mathbb{R}^N$ or $\Omega = \mathbb{R}_+^N$) or $\Omega \subset \mathbb{R}^N$ is an open domain but $0 \notin \bar{\Omega}$. In this section, we are aim to study the existence of nontrivial ground state solution to the system 7.1. Thanks to the fact of that $s_1 = s_2$, we shall obtain further results: the uniqueness of the nontrivial ground state solution; the relationship between the sharp constant $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ and the domain Ω; the existence of infinitely many sign-changing solutions to system 7.1. Finally, we will explore some approaches for studying the sharp constant $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ when Ω is not necessarily a cone.

Noting that for the special case $s_1 = s_2 = s \in (0,2)$ and $\alpha + \beta = 2^*(s)$, the nonlinearities are homogeneous which enable us to define the following constant

$$S_{\alpha,\beta,\lambda,\mu}(\Omega) := \inf_{(u,v)\in\widetilde{\mathscr{D}}} \frac{\int_\Omega \big(|\nabla u|^2 + |\nabla v|^2\big)dx}{\Big(\int_\Omega \big(\lambda\frac{|u|^{2^*(s)}}{|x|^s} + \mu\frac{|v|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa\frac{|u|^\alpha|v|^\beta}{|x|^s}\big)dx\Big)^{\frac{2}{2^*(s)}}}, \tag{7.2}$$

where

$$\widetilde{\mathscr{D}} := \{(u,v)\in\mathscr{D} : \int_\Omega \big(\lambda\frac{|u|^{2^*(s)}}{|x|^s} + \mu\frac{|v|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa\frac{|u|^\alpha|v|^\beta}{|x|^s}\big)dx > 0\}. \tag{7.3}$$

The above constant determines the following kind of inequalities:

$$\begin{aligned} S_{\alpha,\beta,\lambda,\mu}(\Omega)\Big(\int_\Omega \big(\lambda\frac{|u|^{2^*(s)}}{|x|^s} + \mu\frac{|v|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa\frac{|u|^\alpha|v|^\beta}{|x|^s}\big)dx\Big)^{\frac{2}{2^*(s)}} \\ \le \int_\Omega \big(|\nabla u|^2 + |\nabla v|^2\big)dx \end{aligned} \tag{7.4}$$

for $(u,v)\in\mathscr{D}$ or $\widetilde{\mathscr{D}}$, which can be viewed as the double-variable CKN inequality. To the best of our knowledge, such kind of inequality and its sharp constant with extremal functions have not been studied before.

Denote

$$\mu_s(\Omega):=\inf\Big\{\frac{\int_\Omega|\nabla u|^2dx}{(\int_\Omega\frac{|u|^{2^*(s)}}{|x|^s}dx)^{\frac{2}{2^*(s)}}}:\ u\in D_0^{1,2}(\Omega)\backslash\{0\}\Big\},\tag{7.5}$$

then $\mu_s(\Omega)$ can be attained when Ω is a cone in $\mathbb{R}^N$ and $0<s<2$ (see [10]). Noting that $D_0^{1,2}(\Omega)\times\{0\}\subset\widetilde{\mathscr{D}}$ and $\{0\}\times D_0^{1,2}(\Omega)\subset\widetilde{\mathscr{D}}$, by the definition, we have that

$$S_{\alpha,\beta,\lambda,\mu}(\Omega)\leq\big(\max\{\lambda,\mu\}\big)^{-\frac{2}{2^*(s)}}\mu_s(\Omega).\tag{7.6}$$

By Young's inequality, it is easy to see that $S_{\alpha,\beta,\lambda,\mu}(\Omega)>0$. The following statement is obvious.

Proposition 7.1. *Assume that (u,v) is an extremal of $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ such that*

$$\int_\Omega\frac{1}{|x|^s}\big[\lambda|u|^{2^*(s)}+\mu|v|^{2^*(s)}+2^*(s)\kappa|u|^\alpha|v|^\beta\big]dx=1$$

and

$$\int_\Omega\big(|\nabla u|^2+|\nabla v|^2\big)dx=S_{\alpha,\beta,\lambda,\mu}(\Omega).$$

Then

$$(\phi,\psi):=\Big((S_{\alpha,\beta,\lambda,\mu}(\Omega))^{\frac{1}{2^*(s)-2}}u,\ \ (S_{\alpha,\beta,\lambda,\mu}(\Omega))^{\frac{1}{2^*(s)-2}}v\Big)$$

is a ground state solution to problem (7.1) *and the corresponding energy*

$$c_0=\Phi(\phi,\psi)=\big(\frac12-\frac{1}{2^*(s)}\big)\big[S_{\alpha,\beta,\lambda,\mu}(\Omega)\big]^{\frac{2^*(s)}{2^*(s)-2}}.$$

Lemma 7.1. *Assume that $\alpha>0,\beta>0,\lambda>0,\mu>0$, then there exists a best constant*

$$\kappa(\alpha,\beta,\lambda,\mu)=(\alpha+\beta)(\frac{\lambda}{\alpha})^{\frac{\alpha}{\alpha+\beta}}(\frac{\mu}{\beta})^{\frac{\beta}{\alpha+\beta}}\tag{7.7}$$

such that

$$\kappa(\alpha,\beta,\lambda,\mu)\int_\Omega|u|^\alpha|v|^\beta d\nu\leq\lambda\int_\Omega|u|^{\alpha+\beta}d\nu+\mu\int_\Omega|v|^{\alpha+\beta}d\nu$$

for all $(u,v)\in L^{\alpha+\beta}(\Omega,d\nu)\times L^{\alpha+\beta}(\Omega,d\nu)$.

Proof. By Young's inequality with ε,

$$xy\leq\varepsilon x^p+C(\varepsilon)y^q\quad(x,y>0,\varepsilon>0)\tag{7.8}$$

where $\frac{1}{p}+\frac{1}{q}=1, C(\varepsilon)=(\varepsilon p)^{-q/p}q^{-1}$. Take $p=\frac{\alpha+\beta}{\alpha}, q=\frac{\alpha+\beta}{\beta}$ and

$$x=|u|^{\alpha}, y=|v|^{\beta}, \varepsilon=\frac{1}{\alpha+\beta}(\frac{\lambda}{\mu})^{\frac{\beta}{\alpha+\beta}}\alpha^{\frac{\alpha}{\alpha+\beta}}\beta^{\frac{\beta}{\alpha+\beta}},$$

then we obtain that

$$|u|^{\alpha}|v|^{\beta}\leq\frac{1}{\kappa(\alpha,\beta,\lambda,\mu)}\big(\lambda|u|^{\alpha+\beta}+\mu|v|^{\alpha+\beta}\big). \tag{7.9}$$

Hence, for all $(u,v)\in L^{\alpha+\beta}(\Omega,d\nu)\times L^{\alpha+\beta}(\Omega,d\nu)$, we have

$$\kappa(\alpha,\beta,\lambda,\mu)\int_{\Omega}|u|^{\alpha}|v|^{\beta}d\nu\leq\lambda\int_{\Omega}|u|^{\alpha+\beta}d\nu+\mu\int_{\Omega}|v|^{\alpha+\beta}d\nu. \tag{7.10}$$

And we note that when (u,tu) with $t=(\frac{\lambda\beta}{\mu\alpha})^{\frac{1}{\alpha+\beta}}$, the constant $\kappa(\alpha,\beta,\lambda,\mu)$ is attained. Hence, $\kappa(\alpha,\beta,\lambda,\mu)$ is the best constant. □

Lemma 7.2. *Assume $\kappa\leq 0$. Then*

$$S_{\alpha,\beta,\lambda,\mu}(\Omega)=\big(\max\{\lambda,\mu\}\big)^{-\frac{2}{2^*(s)}}\mu_s(\Omega).$$

In particular, in this case $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ can only be attained by the following semi-trivial pairs:

$$\begin{cases}(U,0) & \text{if } \lambda>\mu;\\ (0,U) & \text{if } \lambda<\mu;\\ (U,0) \text{ or } (0,U) & \text{if } \lambda=\mu;\end{cases}$$

where U is an extremal function of $\mu_s(\Omega)$. Hence, the ground state to problem (7.1) *can only be attained by semi-trivial pairs.*

Proof. By (7.6), we only need to prove the reverse inequality. Indeed, for any $(u,v)\in\tilde{\mathscr{D}}$, we have

$$\begin{aligned}&\frac{\int_{\Omega}\big(|\nabla u|^2+|\nabla v|^2\big)dx}{\Big(\int_{\Omega}\big(\lambda\frac{|u|^{2^*(s)}}{|x|^s}+\mu\frac{|v|^{2^*(s)}}{|x|^s}+2^*(s)\kappa\frac{|u|^{\alpha}|v|^{\beta}}{|x|^s}\big)dx\Big)^{\frac{2}{2^*(s)}}}\\
\geq&\frac{\int_{\Omega}\big(|\nabla u|^2+|\nabla v|^2\big)dx}{\Big(\int_{\Omega}\big(\lambda\frac{|u|^{2^*(s)}}{|x|^s}+\mu\frac{|v|^{2^*(s)}}{|x|^s}\Big)^{\frac{2}{2^*(s)}}}\\
\geq&\big(\max\{\lambda,\mu\}\big)^{-\frac{2}{2^*(s)}}\frac{\int_{\Omega}\big(|\nabla u|^2+|\nabla v|^2\big)dx}{\Big(\int_{\Omega}\big(\frac{|u|^{2^*(s)}}{|x|^s}+\frac{|v|^{2^*(s)}}{|x|^s}\big)dx\Big)^{\frac{2}{2^*(s)}}}\\
\geq&\big(\max\{\lambda,\mu\}\big)^{-\frac{2}{2^*(s)}}\mu_s(\Omega)\frac{\big(\int_{\Omega}\frac{|u|^{2^*(s)}}{|x|^s}dx\big)^{\frac{2}{2^*(s)}}+\big(\int_{\Omega}\frac{|v|^{2^*(s)}}{|x|^s}dx\big)^{\frac{2}{2^*(s)}}}{\big(\int_{\Omega}\frac{|u|^{2^*(s)}}{|x|^s}+\frac{|v|^{2^*(s)}}{|x|^s}dx\big)^{\frac{2}{2^*(s)}}}\\
\geq&\big(\max\{\lambda,\mu\}\big)^{-\frac{2}{2^*(s)}}\mu_s(\Omega).\end{aligned} \tag{7.11}$$

By taking the infimum over $\tilde{\mathscr{D}}$, we obtain that

$$S_{\alpha,\beta,\lambda,\mu}(\Omega) \geq \big(\max\{\lambda, \mu\}\big)^{-\frac{2}{2^*(s)}} \mu_s(\Omega).$$

Moreover, by the processes of (7.11), we see that $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ is only achieved by

$$\begin{cases} (U,0) & \text{if } \lambda > \mu; \\ (0,U) & \text{if } \lambda < \mu; \\ (U,0) \text{ or } (0,U) & \text{if } \lambda = \mu, \end{cases}$$

where U is an extremal function of $\mu_s(\Omega)$. □

Proposition 7.2. *$\widetilde{\mathscr{D}} = \mathscr{D}\backslash\{(0,0)\}$ when $\kappa > -(\dfrac{\lambda}{\alpha})^{\frac{\alpha}{2^*(s)}}(\dfrac{\mu}{\beta})^{\frac{\beta}{2^*(s)}}$.*

Proof. By Lemma 7.1, if $\kappa > -(\dfrac{\lambda}{\alpha})^{\frac{\alpha}{2^*(s)}}(\dfrac{\mu}{\beta})^{\frac{\beta}{2^*(s)}}$, then there exists some $C > 0$ such that

$$\int_\Omega \big(\lambda\frac{|u|^{2^*(s)}}{|x|^s} + \mu\frac{|v|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa\frac{|u|^\alpha|v|^\beta}{|x|^s}\big)dx \geq C\Big(\int_\Omega(\lambda\frac{|u|^{2^*(s)}}{|x|^s} + \mu\frac{|v|^{2^*(s)}}{|x|^s})dx\Big). \tag{7.12}$$

Thereby this proposition is proved. □

Remark 7.1. *By the Lemma 7.2 and Proposition 7.2, we know that, when $0 > \kappa > -(\frac{\lambda}{\alpha})^{\frac{\alpha}{2^*(s)}}(\frac{\mu}{\beta})^{\frac{\beta}{2^*(s)}}$, then the inequality (7.4) is meaningful but the sharp constant $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ can be reached only by semi-trivial extremals.*

Note: In view of Lemma 7.2, Proposition 7.2 and Remark 7.1, we have to consider the case that $\kappa > 0$. Therefore, throughout the remaining part of the current paper, we assume $\kappa > 0$.

We obtain the following result:

Theorem 7.1. *Assume Ω is a cone in $\mathbb{R}^N$(especially, $\Omega = \mathbb{R}^N$ or $\Omega = \mathbb{R}^N_+$) or Ω is an open domain with $0 \notin \bar{\Omega}$. If $0 < s < 2, \alpha > 1, \beta > 1, \alpha + \beta = 2^*(s)$ and $\kappa > 0$, then $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ is always achieved by some nonnegative function (u, v).*

Remark 7.2. *Theorem 7.1 asserts that the constant $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ can be attained by a nonnegative extremal function. But at this moment, we can not exclude the possibility of the semi-triviality of the extremal function. Fortunately, we will see further results in Theorem 7.2 below for the nontrivial extremal function.*

When we study the critical problem of a scalar equation, say (P), blow-up analysis is one of the classical method. Its ideas usually are as follows: Consider a modified

subcritical problem define on a convex domain, say (P_ε), which approximates the problem (P). Then study the existence of positive solution u_ε to the modified problem (P_ε) and the regularity of the positive solutions. By the standard Pohozaev identity, u_ε must blow-up as $\varepsilon \to 0$. Apply the standard blow-up arguments to obtain the existence of positive solution of (P). It follows that the set of solutions is nonempty. Therefore, one may take a minimizing sequence to approach the ground state. In this process, the approximation involves the domains and the format of the nonlinearities. However, for the system 7.1, the customary skills can not be applied directly. Since we want to get rid of the semi-trivial solution, the main obstacles lies in that we can not get the precise estimate about the limit of the least energy of the approximate problem. Therefore, in the current paper, we introduce a new approximation system where we just modify the singularity.

7.1 Approximating the problems

For any $\varepsilon \geq 0$, set

$$a_\varepsilon(x) := \begin{cases} \frac{1}{|x|^{s-\varepsilon}} & \text{for } |x| < 1, \\ \frac{1}{|x|^{s+\varepsilon}} & \text{for } |x| \geq 1. \end{cases} \tag{7.13}$$

Lemma 7.3. *Let $\varepsilon > 0$. Then for any $u \in D_0^{1,2}(\Omega)$, $\int_\Omega a_\varepsilon(x)|u|^{2^*(s)}dx$ is well defined and decreasing by ε.*

Proof. Let $\varepsilon_1 > \varepsilon_2 \geq 0$. By the definition of $a_\varepsilon(x)$, it is easy to obtain the result by noting that $a_{\varepsilon_1}(x) < a_{\varepsilon_2}(x) \leq a_0(x)$. □

We also note that for any compact set $\Omega_1 \subset \Omega$ such that $0 \notin \bar{\Omega}_1$, $a_\varepsilon(x) \to a_0(x)$ uniformly on Ω_1 as $\varepsilon \to 0$. For any fixed $\varepsilon > 0$, we consider the ground state solution to the following problem:

$$\begin{cases} -\Delta u - \lambda a_\varepsilon(x)|u|^{2^*(s)-2}u = \kappa\alpha a_\varepsilon(x)|u|^{\alpha-2}u|v|^\beta & \text{in } \Omega, \\ -\Delta v - \mu a_\varepsilon(x)|v|^{2^*(s)-2}v = \kappa\beta a_\varepsilon(x)|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ \kappa > 0, (u,v) \in \mathscr{D} := D_0^{1,2}(\Omega) \times D_0^{1,2}(\Omega), \end{cases} \tag{7.14}$$

Consider the following variational problem

$$\min \int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right)dx \tag{7.15}$$

$$\text{s.t.} \quad \int_\Omega a_\varepsilon(x)\left[\lambda|u|^{2^*(s)} + \mu|v|^{2^*(s)} + 2^*(s)\kappa|u|^\alpha|v|^\beta\right]dx = 1. \tag{7.16}$$

We let $S^\varepsilon_{\alpha,\beta,\lambda,\mu}(\Omega)$ be the minimize value of (7.15), then we have:

Lemma 7.4. *The constant $S^\varepsilon_{\alpha,\beta,\lambda,\mu}(\Omega)$ is increasing with respect to ε and*

$$\lim_{\varepsilon \to 0^+} S^\varepsilon_{\alpha,\beta,\lambda,\mu}(\Omega) = S_{\alpha,\beta,\lambda,\mu}(\Omega).$$

Proof. By the definition of $S^{\varepsilon}_{\alpha,\beta,\lambda,\mu}(\Omega)$ and $a_\varepsilon(x)$, it is easy to see that

$$S^{\varepsilon_1}_{\alpha,\beta,\lambda,\mu}(\Omega) \geq S^{\varepsilon_2}_{\alpha,\beta,\lambda,\mu}(\Omega) \text{ for aly } \varepsilon_1 > \varepsilon_2 \geq 0.$$

Hence, we have

$$\liminf_{\varepsilon\to 0^+} S^{\varepsilon}_{\alpha,\beta,\lambda,\mu}(\Omega) \geq S_{\alpha,\beta,\lambda,\mu}(\Omega). \tag{7.17}$$

On the other hand, for any $\eta > 0$, there exists $(u,v) \in C_c^\infty(\Omega) \times C_c^\infty(\Omega)$ such that

$$\int_\Omega a_0(x)\big[\lambda|u|^{2^*(s)} + \mu|v|^{2^*(s)} + 2^*(s)\kappa|u|^\alpha|v|^\beta\big]dx = 1$$

and

$$\int_\Omega \big(|\nabla u|^2 + |\nabla v|^2\big)dx < S_{\alpha,\beta,\lambda,\mu}(\Omega) + \eta.$$

Since $a_\varepsilon(x) \to a_0(x)$ in $L^\infty(supp(u))$ as $\varepsilon \to 0$, we obtain that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0^+} S^{\varepsilon}_{\alpha,\beta,\lambda,\mu}(\Omega)\\
=&\limsup_{\varepsilon\to 0^+} \frac{\int_\Omega \big(|\nabla u|^2 + |\nabla v|^2\big)dx}{\Big(\int_\Omega a_\varepsilon(x)\big[\lambda|u|^{2^*(s)} + \mu|v|^{2^*(s)} + 2^*(s)\kappa|u|^\alpha|v|^\beta\big]dx\Big)^{\frac{2}{2^*(s)}}}\\
=&\frac{\int_\Omega \big(|\nabla u|^2 + |\nabla v|^2\big)dx}{\Big(\int_\Omega a_0(x)\big[\lambda|u|^{2^*(s)} + \mu|v|^{2^*(s)} + 2^*(s)\kappa|u|^\alpha|v|^\beta\big]dx\Big)^{\frac{2}{2^*(s)}}}\\
=&\int_\Omega \big(|\nabla u|^2 + |\nabla v|^2\big)dx\\
<&S_{\alpha,\beta,\lambda,\mu}(\Omega) + \eta.
\end{aligned}$$

By the arbitrariness of η, we have

$$\limsup_{\varepsilon\to 0^+} S^{\varepsilon}_{\alpha,\beta,\lambda,\mu}(\Omega) \leq S_{\alpha,\beta,\lambda,\mu}(\Omega). \tag{7.18}$$

Thus the proof is completed by (7.17) and (7.18). $\square$

Let $L^p(\Omega, a_\varepsilon(x)dx)$ denote the space of L^p-integrable functions with respect to the measure $a_\varepsilon(x)dx$ and the corresponding norm is indicated by

$$|u|_{p,\varepsilon} := \Big(\int_\Omega a_\varepsilon(x)|u|^p dx\Big)^{\frac{1}{p}}, \quad p > 1.$$

Then we have the following compact embedding result:

Lemma 7.5. *For any $\varepsilon \in (0,s)$, the embedding $D_0^{1,2}(\Omega) \hookrightarrow L^{2^*(s)}(\Omega, a_\varepsilon(x)dx)$ is compact.*

Proof. Let $\{u_n\} \subset D_0^{1,2}(\Omega)$ be a bounded sequence. Up to a subsequence, we may assume that $u_n \rightharpoonup u$ in $D_0^{1,2}(\Omega)$ and $u_n \to u$ a.e. in Ω. Then for any $R > 1$, we have

$$\int_{\Omega \cap B_R^c} a_\varepsilon(x)|u|^{2^*(s)}dx \le \frac{1}{R^\varepsilon}\int_{\Omega \cap B_R^c} a_0|u|^{2^*(s)}dx \to 0, \tag{7.19}$$

uniformly for all n as $R \to +\infty$.

Noting that $2^*(s) < 2^*(s-\varepsilon), 2^*(s) < 2^* := \frac{2N}{N-2}$, by Rellich-Kondrachov compact theorem, we have

$$\lim_{n\to\infty}\int_{\Omega \cap B_R} a_\varepsilon(x)|u_n - u|^{2^*(s)}dx = 0. \tag{7.20}$$

By (7.19) and (7.20), we prove this Lemma. □

Lemma 7.6. *For any* $\varepsilon \in (0, s)$, $S^\varepsilon_{\alpha,\beta,\lambda,\mu}(\Omega)$ *is attained by some extremal* $(u_\varepsilon, v_\varepsilon)$, *i.e.,*

$$\int_\Omega a_\varepsilon(x)\big[\lambda|u_\varepsilon|^{2^*(s)} + \mu|v_\varepsilon|^{2^*(s)} + 2^*(s)\kappa|u_\varepsilon|^\alpha|v_\varepsilon|^\beta\big]dx = 1$$

and

$$\int_\Omega \big(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2\big)dx = S^\varepsilon_{\alpha,\beta,\lambda,\mu}(\Omega).$$

Moreover, $(u_\varepsilon, v_\varepsilon)$ *satisfies the following equation:*

$$\begin{cases} -\Delta u = S^\varepsilon_{\alpha,\beta,\lambda,\mu}(\Omega)\Big(\lambda a_\varepsilon(x)|u|^{2^*(s)-2}u + \kappa\alpha a_\varepsilon(x)|u|^{\alpha-2}u|v|^\beta\Big) & \textit{in } \Omega,\\ -\Delta v = S^\varepsilon_{\alpha,\beta,\lambda,\mu}(\Omega)\Big(\mu a_\varepsilon(x)|v|^{2^*(s)-2}v + \kappa\beta a_\varepsilon(x)|u|^\alpha|v|^{\beta-2}v\Big) & \textit{in } \Omega,\\ u \ge 0, v \ge 0, (u,v) \in \mathscr{D} := D_0^{1,2}(\Omega)\times D_0^{1,2}(\Omega), \end{cases} \tag{7.21}$$

Proof. Let $\{u_{n,\varepsilon}\} \subset \mathscr{D}$ be a minimizing sequence, i.e.,

$$\int_\Omega a_\varepsilon(x)\big[\lambda|u_{n,\varepsilon}|^{2^*(s)} + \mu|v_{n,\varepsilon}|^{2^*(s)} + 2^*(s)\kappa|u_{n,\varepsilon}|^\alpha|v_{n,\varepsilon}|^\beta\big]dx = 1$$

and

$$\int_\Omega \big(|\nabla u_{n,\varepsilon}|^2 + |\nabla u_{n,\varepsilon}|^2\big)dx \to S^\varepsilon_{\alpha,\beta,\lambda,\mu}(\Omega) \text{ as } n \to +\infty.$$

Then we see that $\{u_{n,\varepsilon}\}$ and $\{v_{n,\varepsilon}\}$ are bounded in $D_0^{1,2}(\Omega)$. By Lemma 7.5, we see that up to a subsequence $u_{n,\varepsilon} \to u_\varepsilon$ and $v_{n,\varepsilon} \to v_\varepsilon$ in $L^{2^*(s)}(\Omega, a_\varepsilon(x)dx)$. Hence

$$\int_\Omega a_\varepsilon(x)\big[\lambda|u_\varepsilon|^{2^*(s)} + \mu|v_\varepsilon|^{2^*(s)} + 2^*(s)\kappa|u_\varepsilon|^\alpha|v_\varepsilon|^\beta\big]dx = 1.$$

Then by the definition of $S^\varepsilon_{\alpha,\beta,\lambda,\mu}(\Omega)$, we have

$$\int_\Omega \big(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2\big)dx \ge S^\varepsilon_{\alpha,\beta,\lambda,\mu}(\Omega).$$

On the other hand, by the weak semi-continuous of a norm (or Fatou's Lemma), we have

$$\int_\Omega \big(|\nabla u_\varepsilon|^2+|\nabla v_\varepsilon|^2\big)dx\le \liminf_{n\to\infty}\int_\Omega \big(|\nabla u_{n,\varepsilon}|^2+|\nabla u_{n,\varepsilon}|^2\big)dx=S^\varepsilon_{\alpha,\beta,\lambda,\mu}(\Omega).$$

Hence, $(u_\varepsilon,v_\varepsilon)$ is a minimizer of $S^\varepsilon_{\alpha,\beta,\lambda,\mu}(\Omega)$. Without loss of generality, we may assume that $u_{n,\varepsilon}\ge 0$ and $v_{n,\varepsilon}\ge 0$ for all n. Then since $u_{n,\varepsilon}\to u_\varepsilon, v_{n,\varepsilon}\to v_\varepsilon$ a.e. in Ω, we obtain that $u_\varepsilon\ge 0, v_\varepsilon\ge 0$ a.e. in Ω. There exists some Lagrange multiplier $\Lambda\in\mathbb{R}$ such that

$$\begin{cases}-\Delta u_\varepsilon=\Lambda\Big(\lambda a_\varepsilon(x)|u_\varepsilon|^{2^*(s)-2}u_\varepsilon+\kappa\alpha a_\varepsilon(x)|u_\varepsilon|^{\alpha-2}u_\varepsilon|v_\varepsilon|^\beta\Big) & \text{in }\Omega,\\ -\Delta v_\varepsilon=\Lambda\Big(\mu a_\varepsilon(x)|v_\varepsilon|^{2^*(s)-2}v_\varepsilon+\kappa\beta a_\varepsilon(x)|u_\varepsilon|^{\alpha}|v_\varepsilon|^{\beta-2}v_\varepsilon\Big) & \text{in }\Omega.\end{cases}$$

Testing by $(u_\varepsilon,v_\varepsilon)$, we obtain that $\Lambda=S^\varepsilon_{\alpha,\beta,\lambda,\mu}(\Omega)$. □

Lemma 7.7. *For $\varepsilon\in(0,s)$, assume $(u_\varepsilon,v_\varepsilon)$ is a solution to* (7.21) *given by Lemma 7.6. Then:*

(i) The family $(u_\varepsilon,v_\varepsilon)$ is bounded in $\mathscr{D}$;

(ii) Up to a subsequence, we have $u_\varepsilon\rightharpoonup u, v_\varepsilon\rightharpoonup v$ in $D^{1,2}_0(\Omega)$ and $u_\varepsilon\to u, v_\varepsilon\to v$ a.e. in Ω as $\varepsilon\to 0$.

(iii) (u,v) given in (ii) weakly solves

$$\begin{cases}-\Delta u=S_{\alpha,\beta,\lambda,\mu}(\Omega)\Big(\lambda a_0(x)|u|^{2^*(s)-2}u+\kappa\alpha a_0(x)|u|^{\alpha-2}u|v|^\beta\Big) & \text{in }\Omega,\\ -\Delta v=S_{\alpha,\beta,\lambda,\mu}(\Omega)\Big(\mu a_0(x)|v|^{2^*(s)-2}v+\kappa\beta a_0(x)|u|^{\alpha}|v|^{\beta-2}v\Big) & \text{in }\Omega,\\ u\ge 0, v\ge 0, (u,v)\in\mathscr{D},\end{cases}\tag{7.22}$$

(iv) If $(u,v)\neq(0,0)$, then

$$\int_\Omega a_0(x)\big[\lambda|u|^{2^*(s)}+\mu|v|^{2^*(s)}+2^*(s)\kappa|u|^\alpha|v|^\beta\big]dx=1$$

and $u_\varepsilon\to u$ and $v_\varepsilon\to v$ strongly in $D^{1,2}_0(\Omega)$. Moreover, (u,v) is an extremal function of $S_{\alpha,\beta,\lambda,\mu}(\Omega)$.

Proof. (i) follows by Lemma 7.4 and (ii) is trivial;

(iii) Without loss of generality, we assume that $\varepsilon_k\downarrow 0$ as $k\to\infty$. For any $\phi\in C_c^\infty(\Omega),\psi\in C_c^\infty(\Omega)$, since $(u_{\varepsilon_k},v_{\varepsilon_k})$ is a solution of (7.21) with $\varepsilon=\varepsilon_k$, we have

$$\begin{aligned}&\int_\Omega\big(\nabla u_{\varepsilon_k}\cdot\nabla\phi+\nabla v_{\varepsilon_k}\cdot\nabla\psi\big)dx\\ &=S^{\varepsilon_k}_{\alpha,\beta,\lambda,\mu}(\Omega)\int_\Omega\Big(\lambda a_\varepsilon(x)|u_\varepsilon|^{2^*(s)-2}u_\varepsilon\phi+\kappa\alpha a_\varepsilon(x)|u_\varepsilon|^{\alpha-2}u_\varepsilon\phi|v_\varepsilon|^\beta\Big)dx\\ &\quad+S^{\varepsilon_k}_{\alpha,\beta,\lambda,\mu}(\Omega)\int_\Omega\Big(\mu a_\varepsilon(x)|v_\varepsilon|^{2^*(s)-2}v_\varepsilon\psi+\kappa\beta a_\varepsilon(x)|u_\varepsilon|^{\alpha}|v_\varepsilon|^{\beta-2}v_\varepsilon\psi\Big)dx.\end{aligned}\tag{7.23}$$

Recalling that $a_{\varepsilon_k}(x) \to a_0(x)$ in $L^\infty(sppt(\phi)\cup sppt(\psi))$ and $S^{\varepsilon_k}_{\alpha,\beta,\lambda,\mu}(\Omega) \to S_{\alpha,\beta,\lambda,\mu}(\Omega)$ as $k \to \infty$, we obtain that

$$\begin{aligned}
&\int_\Omega \big(\nabla u \cdot \nabla\phi + \nabla v \cdot \nabla\psi\big) dx \\
&= S_{\alpha,\beta,\lambda,\mu}(\Omega) \int_\Omega \Big(\lambda a_0(x)|u|^{2^*(s)-2}u\phi + \kappa\alpha a_0(x)|u|^{\alpha-2}u\phi|v|^\beta\Big) dx \\
&\quad + S_{\alpha,\beta,\lambda,\mu}(\Omega) \int_\Omega \Big(\mu a_0(x)|v|^{2^*(s)-2}v\psi + \kappa\beta a_0(x)|u|^\alpha|v|^{\beta-2}v\psi\Big) dx. \qquad (7.24)
\end{aligned}$$

Since (ϕ,ψ) is arbitrary, we see that (u,v) weakly solve (7.22).

(iv) By Fatou's lemma, we have

$$\begin{aligned}
&\int_\Omega a_0(x)\big[\lambda|u|^{2^*(s)} + \mu|v|^{2^*(s)} + 2^*(s)\kappa|u|^\alpha|v|^\beta\big] dx \\
\le &\liminf_{k\to\infty} \int_\Omega a_{\varepsilon_k}(x)\big[\lambda|u_{\varepsilon_k}|^{2^*(s)} + \mu|v_{\varepsilon_k}|^{2^*(s)} + 2^*(s)\kappa|u_{\varepsilon_k}|^\alpha|v_{\varepsilon_k}|^\beta\big] dx = 1. \qquad (7.25)
\end{aligned}$$

If $\int_\Omega a_0(x)\big[\lambda|u|^{2^*(s)} + \mu|v|^{2^*(s)} + 2^*(s)\kappa|u|^\alpha|v|^\beta\big] dx \ne 1$, since $(u,v) \ne (0,0)$, we have

$$0 < \int_\Omega a_0(x)\big[\lambda|u|^{2^*(s)} + \mu|v|^{2^*(s)} + 2^*(s)\kappa|u|^\alpha|v|^\beta\big] dx < 1.$$

Hence, by (iii) and the definition of $S_{\alpha,\beta,\lambda,\mu}(\Omega)$, we have

$$\begin{aligned}
S_{\alpha,\beta,\lambda,\mu}(\Omega) =& \frac{\int_\Omega \big(|\nabla u|^2 + |\nabla v|^2\big) dx}{\int_\Omega a_0(x)\big[\lambda|u|^{2^*(s)} + \mu|v|^{2^*(s)} + 2^*(s)\kappa|u|^\alpha|v|^\beta\big] dx} \\
>& \frac{\int_\Omega \big(|\nabla u|^2 + |\nabla v|^2\big) dx}{\Big(\int_\Omega a_0(x)\big[\lambda|u|^{2^*(s)} + \mu|v|^{2^*(s)} + 2^*(s)\kappa|u|^\alpha|v|^\beta\big] dx\Big)^{\frac{2}{2^*(s)}}} \\
\ge& S_{\alpha,\beta,\lambda,\mu}(\Omega), \qquad (7.26)
\end{aligned}$$

a contradiction. Hence, $\int_\Omega a_0(x)\big[\lambda|u|^{2^*(s)} + \mu|v|^{2^*(s)} + 2^*(s)\kappa|u|^\alpha|v|^\beta\big] dx = 1$. It follows that

$$\int_\Omega \big(|\nabla u_{\varepsilon_k}|^2 + |\nabla v_{\varepsilon_k}|^2\big) \to \int_\Omega \big(|\nabla u|^2 + |\nabla v|^2\big) dx, \qquad (7.27)$$

which implies that $u_{\varepsilon_k} \to u, v_{\varepsilon_k} \to v$ in $D^{1,2}_0(\Omega)$. □

7.2 Pohozaev Identity and the Proof of Theorem 7.1

Proposition 7.3. *Let $(u,v) \in D^{1,2}_0(\Omega) \times D^{1,2}_0(\Omega)$ be a solution of the system*

$$\begin{cases} -\Delta u &= G_u(x,u,v), \\ -\Delta v &= G_v(x,u,v), \end{cases}$$

where

$$G(x,u,v) = \int_0^u G_s(x,s,v)ds + G(x,0,v) = \int_0^v G_t(x,u,t)dt + G(x,u,0)$$

is such that $G(x,0,0) \equiv 0, G(\cdot,u(\cdot),v(\cdot))$ *and that* $x_i G_{x_i}(\cdot,u(\cdot),v(\cdot))$ *are in* $L^1(\Omega)$. *Then* (u,v) *satisfies:*

$$\begin{aligned}&\int_{\partial\Omega} |\nabla(u,v)|^2 x\cdot\nu d\sigma \\ =&2N\int_\Omega G(x,u,v)dx + 2\sum_{i=1}^N \int_\Omega x_i G_{x_i}(x,u,v)dx - (N-2)\int_\Omega |\nabla(u,v)|^2 dx,\end{aligned} \tag{7.28}$$

where Ω *is a regular domain in* $\mathbb{R}^N$, ν *denotes the unitary exterior normal vector to* $\partial\Omega$ *and* $|\nabla(u,v)|^2 := |\nabla u|^2 + |\nabla v|^2$. *Moreover, if* $\Omega = \mathbb{R}^N$ *or a cone, then*

$$2N\int_\Omega G(x,u,v)dx + 2\sum_{i=1}^N \int_\Omega x_i G_{x_i}(x,u,v)dx = (N-2)\int_\Omega |\nabla(u,v)|^2 dx. \tag{7.29}$$

Proof. Since (u,v) is a solution, then we have

$$0 = \big(-\Delta u + G_u(x,u,v)\big)x\cdot\nabla u = \big(-\Delta v + G_v(x,u,v)\big)x\cdot\nabla v. \tag{7.30}$$

It is clear that

$$-\Delta u x\cdot\nabla u = -div(\nabla u x\cdot\nabla u - x\frac{|\nabla u|^2}{2}) - \frac{N-2}{2}|\nabla u|^2,$$

$$-\Delta v x\cdot\nabla v = -div(\nabla v x\cdot\nabla v - x\frac{|\nabla v|^2}{2}) - \frac{N-2}{2}|\nabla v|^2,$$

$$\begin{aligned}&G_u(x,u,v)x\cdot\nabla u + G_v(x,u,v)x\cdot\nabla v \\ &= div(xG(x,u,v)) - NG(x,u,v) - \sum_{i=1}^N x_i G_{x_i}(x,u,v).\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}&\int_{\partial\Omega}\big(\sigma G(\sigma,u,v) + (\nabla u\sigma\cdot\nabla u - \sigma\frac{|\nabla u|^2}{2}) + (\nabla v\sigma\cdot\nabla v - \sigma\frac{|\nabla v|^2}{2})\big)\cdot\nu d\sigma \\ =&\int_\Omega \Big(NG(x,u,v) - \frac{N-2}{2}|\nabla(u,v)|^2 + \sum_{i=1}^N x_i G_{x_i}(x,u,v)\Big)dx.\end{aligned} \tag{7.31}$$

When $u = v = 0$ on $\partial\Omega$, we have

$$\nabla u = \nabla u\cdot\nu\nu, \nabla v = \nabla v\cdot\nu\nu. \tag{7.32}$$

Then by (7.31), it follows that

$$\begin{aligned}&\int_{\partial\Omega}\big(G(\sigma,u,v)+\frac{1}{2}|\nabla(u,v)|^2\big)\sigma\cdot\nu d\sigma\\ &=\int_{\Omega}\big(NG(x,u,v)-\frac{N-2}{2}|\nabla(u,v)|^2+\sum_{i=1}^{N}x_iG_{x_i}(x,u,v)\big)dx.\end{aligned}\tag{7.33}$$

Moreover, since $G(x,0,0)\equiv 0$, if $u=v=0$ on $\partial\Omega$, we have $G(\sigma,u,v)\equiv 0$ on $\partial\Omega$, then we obtain that

$$\frac{1}{2}\int_{\partial\Omega}|\nabla(u,v)|^2\sigma\cdot\nu d\sigma=\int_{\Omega}\big(NG(x,u,v)-\frac{N-2}{2}|\nabla(u,v)|^2+\sum_{i=1}^{N}x_iG_{x_i}(x,u,v)\big)dx,\tag{7.34}$$

which is equivalent to (7.28). Using polar coordinate transformation, since $|\nabla(u,v)|\in L^2(\mathbb{R}^N)$, we have

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla(u,v)|^2dx&=\int_{\mathbb{R}^N}|\nabla u|^2+|\nabla v|^2dx\\ &=\int_0^\infty\int_{\partial B_r(0)}|\nabla u(r,\theta)|^2+|\nabla v(r,\theta)|^2d\theta r^{N-1}dr\\ &=\int_0^\infty\zeta(r)dr<\infty,\end{aligned}$$

where $\zeta(r):=\int_{\partial B_r(0)}|\nabla u(r,\theta)|^2+|\nabla v(r,\theta)|^2d\theta r^{N-1}\geq 0$, then by the absolute continuity, there exists $r_n\to\infty$ such that $\zeta(r_n)\to 0$. Since $N\geq 3$, we have

$$\int_{\partial B_{r_n}(0)}\big(|\nabla u(r_n,\theta)|^2+|\nabla v(r_n,\theta)|^2\big)r_nd\theta\to 0,$$

which implies that

$$\int_{\partial B_R(0)}|\nabla(u,v)|^2\sigma\cdot\nu d\sigma\to 0 \text{ as } R\to\infty.\tag{7.35}$$

Since $|G(x,u.,v)|\in L^1(\mathbb{R}^N)$, then by the the similar arguments, we obtain that

$$\int_{\partial B_R(0)}G(\sigma,u,v)\sigma\cdot\nu d\sigma\to 0 \text{ as } R\to\infty.\tag{7.36}$$

When considering $\Omega=B_R(0)$ in (7.31), it follows that $\nu=\frac{x}{|x|},\ \sigma\cdot\nu=|x|,\ 0\leq(\nabla u\sigma\cdot\nabla u)\cdot\nu\leq|\nabla u|^2\sigma\cdot\nu$. Hence by (7.35), we have

$$\int_{\partial B_R(0)}(\nabla u\sigma\cdot\nabla u-\sigma\frac{|\nabla u|^2}{2})\cdot\nu d\sigma\to 0 \text{ as } R\to+\infty.\tag{7.37}$$

Similarly, we also have

$$\int_{\partial B_R(0)}(\nabla v\sigma\cdot\nabla v-\sigma\frac{|\nabla v|^2}{2})\cdot\nu d\sigma\to 0 \text{ as } R\to+\infty.\tag{7.38}$$

Finally, by the Lebesgue dominated convergence theorem, when $R \to \infty$, we have that

$$\int_{B_R(0)} |\nabla(u,v)|^2 dx \to \int_{\mathbb{R}^N} |\nabla(u,v)|^2 dx, \int_{B_R(0)} G(x,u,v)dx \to \int_{\mathbb{R}^N} G(x,u,v)dx,$$

and

$$\sum_{i=1}^N \int_{B_R(0)} x_i G_{x_i}(x,u,v)dx \to \sum_{i=1}^N \int_{\mathbb{R}^N} x_i G_{x_i}(x,u,v)dx.$$

Combining with these results and (7.31), (7.36), (7.37), (7.38) we obtain (7.29). The case of that Ω is a cone, we have $x \cdot \nu \equiv 0$ for $x \in \partial\Omega$, then (7.29) follows by (7.28) easily. $\square$

Corollary 7.1. *Let* $0 < \varepsilon < s < 2$ *and* $a_\varepsilon(x)$ *be defined by* (7.13). *Suppose that* $\alpha > 1, \beta > 1, \alpha + \beta = 2^*(s)$ *and* Ω *is a cone. Then any solution* (u,v) *of*

$$\begin{cases} -\Delta u = S^\varepsilon_{\alpha,\beta,\lambda,\mu}(\Omega)\Big(\lambda a_\varepsilon(x)|u|^{2^*(s)-2}u + \kappa\alpha a_\varepsilon(x)|u|^{\alpha-2}u|v|^\beta\Big) & \textit{in } \Omega, \\ -\Delta v = S^\varepsilon_{\alpha,\beta,\lambda,\mu}(\Omega)\Big(\mu a_\varepsilon(x)|v|^{2^*(s)-2}v + \kappa\beta a_\varepsilon(x)|u|^\alpha|v|^{\beta-2}v\Big) & \textit{in } \Omega, \\ u \geq 0, v \geq 0, (u,v) \in \mathscr{D} := D^{1,2}_0(\Omega) \times D^{1,2}_0(\Omega), \end{cases} \tag{7.39}$$

satisfies

$$\begin{aligned} &\int_{\Omega\cap B_1} \big[\frac{\lambda}{2^*(s)} a_\varepsilon(x)|u|^{2^*(s)} + \frac{\mu}{2^*(s)} a_\varepsilon(x)|v|^{2^*(s)} + 2^*(s)\kappa a_\varepsilon(x)|u|^\alpha|v|^\beta\big]dx \\ =&\int_{\Omega\cap B_1^c} \big[\frac{\lambda}{2^*(s)} a_\varepsilon(x)|u|^{2^*(s)} + \frac{\mu}{2^*(s)} a_\varepsilon(x)|v|^{2^*(s)} + 2^*(s)\kappa a_\varepsilon(x)|u|^\alpha|v|^\beta\big]dx. \end{aligned} \tag{7.40}$$

Proof. Let

$$G(x,u,v) := S^\varepsilon_{\alpha,\beta,\lambda,\mu}(\Omega)\big[\frac{\lambda}{2^*(s)} a_\varepsilon(x)|u|^{2^*(s)} + \frac{\mu}{2^*(s)} a_\varepsilon(x)|v|^{2^*(s)} + \kappa a_\varepsilon(x)|u|^\alpha|v|^\beta\big]. \tag{7.41}$$

Noting that

$$\frac{\partial}{\partial x_i} a_\varepsilon(x) = \begin{cases} -(s-\varepsilon)\frac{1}{|x|^{s+2-\varepsilon}} x_i & \text{for } |x| < 1, \\ -(s+\varepsilon)\frac{1}{|x|^{s+2+\varepsilon}} x_i & \text{for } |x| > 1, \end{cases} \tag{7.42}$$

we have

$$
\begin{aligned}
& x_i \cdot G_{x_i}(x,u,v) && (7.43)\\
&= S^{\varepsilon}_{\alpha,\beta,\lambda,\mu}(\Omega)\big[\frac{\lambda}{2^*(s)}|u|^{2^*(s)} + \frac{\mu}{2^*(s)}|v|^{2^*(s)} + \kappa|u|^{\alpha}|v|^{\beta}\big] x_i \frac{\partial}{\partial x_i} a_\varepsilon(x)\\
&= \begin{cases} -(s-\varepsilon) S^{\varepsilon}_{\alpha,\beta,\lambda,\mu}(\Omega)\big[\frac{\lambda}{2^*(s)}|u|^{2^*(s)} + \frac{\mu}{2^*(s)}|v|^{2^*(s)} + \kappa|u|^{\alpha}|v|^{\beta}\big]\frac{1}{|x|^{s+2-\varepsilon}} x_i^2 \\ \text{if } |x|<1, \\ -(s+\varepsilon) S^{\varepsilon}_{\alpha,\beta,\lambda,\mu}(\Omega)\big[\frac{\lambda}{2^*(s)}|u|^{2^*(s)} + \frac{\mu}{2^*(s)}|v|^{2^*(s)} + \kappa|u|^{\alpha}|v|^{\beta}\big]\frac{1}{|x|^{s+2+\varepsilon}} x_i^2 \\ \text{if } |x|>1. \end{cases} && (7.44)
\end{aligned}
$$

Hence, by Proposition 7.3, we have

$$
\begin{aligned}
& -2(N-s)\int_\Omega S^{\varepsilon}_{\alpha,\beta,\lambda,\mu}(\Omega)\big[\frac{\lambda}{2^*(s)} a_\varepsilon(x)|u|^{2^*(s)} && (7.45)\\
& + \frac{\mu}{2^*(s)} a_\varepsilon(x)|v|^{2^*(s)} + \kappa a_\varepsilon(x)|u|^{\alpha}|v|^{\beta}\big]dx\\
& + (N-2)\int_\Omega \big(|\nabla u|^2 + |\nabla v|^2\big)dx\\
& = 2\varepsilon \int_{\Omega\cap B_1} S^{\varepsilon}_{\alpha,\beta,\lambda,\mu}(\Omega)\big[\frac{\lambda}{2^*(s)} a_\varepsilon(x)|u|^{2^*(s)} && (7.46)\\
& \quad + \frac{\mu}{2^*(s)} a_\varepsilon(x)|v|^{2^*(s)} + \kappa a_\varepsilon(x)|u|^{\alpha}|v|^{\beta}\big]dx\\
& \quad - 2\varepsilon \int_{\Omega\cap B_1^c} S^{\varepsilon}_{\alpha,\beta,\lambda,\mu}(\Omega)\big[\frac{\lambda}{2^*(s)} a_\varepsilon(x)|u|^{2^*(s)} && (7.47)\\
& \quad + \frac{\mu}{2^*(s)} a_\varepsilon(x)|v|^{2^*(s)} + \kappa a_\varepsilon(x)|u|^{\alpha}|v|^{\beta}\big]dx. && (7.48)
\end{aligned}
$$

On the other hand, since (u,v) is a solution, we have

$$
\begin{aligned}
& \int_\Omega \big(|\nabla u|^2 + |\nabla v|^2\big)dx\\
& = S^{\varepsilon}_{\alpha,\beta,\lambda,\mu}(\Omega)\int_\Omega \big[\lambda a_\varepsilon(x)|u|^{2^*(s)} + \mu a_\varepsilon(x)|v|^{2^*(s)} + 2^*(s)\kappa a_\varepsilon(x)|u|^{\alpha}|v|^{\beta}\big]dx. && (7.49)
\end{aligned}
$$

Recalling that $S^{\varepsilon}_{\alpha,\beta,\lambda,\mu}(\Omega) > 0, \varepsilon > 0$, by (7.45) and (7.49), we obtain (7.40). □

Corollary 7.2. *Let* $0 < \varepsilon < s < 2$ *and* $a_\varepsilon(x)$ *be defined by* (7.13). *Suppose that* $\alpha > 1, \beta > 1, \alpha + \beta = 2^*(s)$ *and* Ω *is a cone. Let* $(u_\varepsilon, v_\varepsilon)$ *be a solution to* (7.21) *given by Lemma 7.6. Then up to a subsequence, there exists some* $(u,v) \in \mathscr{D}$ *such that* $u_\varepsilon \to u, v_\varepsilon \to v$ *strongly in* $D_0^{1,2}(\Omega)$ *as* $\varepsilon \to 0$.

Proof. By Lemma 7.7, we only need to prove that $(u,v) \neq (0,0)$. Now,we proceed by contradiction. We assume that $u = v = 0$. Let $\chi(x) \in C_c^\infty(\Omega)$ be a cut-off function

such that $\chi(x) \equiv 1$ in $B_{\frac{r}{2}} \cap \Omega$, $\chi(x) \equiv 0$ in $\Omega \backslash B_r$, recalling the Rellich-Kondrachov compact theorem and $2 < 2^*(s) < 2^* := \frac{2N}{N-2}$, we have $u_\varepsilon \to 0$ in $L^t(\Omega_1)$ for all $1 < t < 2^*$ if $0 \notin \bar{\Omega}_1$. Hence, it is easy to see that

$$\begin{aligned}
&\int_\Omega \big[\lambda a_\varepsilon(x)|\chi u_\varepsilon|^{2^*(s)} + \mu a_\varepsilon(x)|\chi v_\varepsilon|^{2^*(s)} + 2^*(s)\kappa a_\varepsilon(x)|\chi u_\varepsilon|^\alpha |\chi v_\varepsilon|^\beta\big]dx \\
&= \int_{\Omega\cap B_r} \big[\lambda a_\varepsilon(x)|u_\varepsilon|^{2^*(s)} + \mu a_\varepsilon(x)|v_\varepsilon|^{2^*(s)} + 2^*(s)\kappa a_\varepsilon(x)|u_\varepsilon|^\alpha |v_\varepsilon|^\beta\big]dx + o(1) \\
&=: \eta_r + o(1).
\end{aligned} \tag{7.50}$$

On the other hand, by the triangle inequality, we have

$$\begin{aligned}
&\int_\Omega \big(|\nabla(\chi u_\varepsilon)|^2 + |\nabla(\chi v_\varepsilon)|^2\big)dx \\
=&\int_\Omega \big(|(\nabla\chi)u_\varepsilon + \chi\nabla u_\varepsilon|^2 + |(\nabla\chi)u_\varepsilon + \chi\nabla v_\varepsilon|^2\big)dx \\
\leq&\Bigg(\big(\int_\Omega |(\nabla\chi)|^2 u_\varepsilon^2 dx\big)^{\frac{1}{2}} + \big(\int_\Omega |\chi|^2|\nabla u_\varepsilon|^2 dx\big)^{\frac{1}{2}} \\
&+ \big(\int_\Omega |(\nabla\chi)|^2 v_\varepsilon^2 dx\big)^{\frac{1}{2}} + \big(\int_\Omega |\chi|^2|\nabla v_\varepsilon|^2 dx\big)^{\frac{1}{2}}\Bigg)^2 \\
=&\int_{\Omega\cap B_r} \big(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2\big)dx + o(1) \\
:=&\sigma_r + o(1).
\end{aligned} \tag{7.51}$$

By (7.50) and 7.51, we obtain that

$$S^\varepsilon_{\alpha,\beta,\lambda,\mu}(\Omega)\big(\eta_r + o(1)\big)^{\frac{2}{2^*(s)}} \leq \sigma_r + o(1). \tag{7.52}$$

Similarly, we take $\tilde{\chi}(x) \in C^\infty(\Omega)$ such that $\tilde{\chi}(x) \equiv 0$ in $B_r \cap \Omega$ and $\tilde{\chi} \equiv 1$ in $\Omega \backslash B_{2r}$. Then by repeating the above steps, we obtain that

$$S^\varepsilon_{\alpha,\beta,\lambda,\mu}(\Omega)\big(1 - \eta_r + o(1)\big)^{\frac{2}{2^*(s)}} \leq S^\varepsilon_{\alpha,\beta,\lambda,\mu} - \sigma_r + o(1). \tag{7.53}$$

By (7.52) and (7.53), we deduce that

$$\big(\eta_r + o(1)\big)^{\frac{2}{2^*(s)}} + \big(1 - \eta_r + o(1)\big)^{\frac{2}{2^*(s)}} \leq 1. \tag{7.54}$$

Notice that $h(t) := t^{\frac{2}{2^*(s)}} + (1-t)^{\frac{2}{2^*(s)}}$ satisfying that $\min_{t\in[0,1]} h(t) = 1$ and only achieved by $t = 0$ or $t = 1$. Hence, we obtain that $\eta_r \equiv 0$ or $\eta_r \equiv 1$ for any $r > 0$.

But by Corollary 7.1, for any $\varepsilon \in (0, s)$, we have

$$\begin{aligned}
&\int_{\Omega\cap B_1} \big[\lambda a_\varepsilon(x)|u_\varepsilon|^{2^*(s)} + \mu a_\varepsilon(x)|v_\varepsilon|^{2^*(s)} + 2^*(s)\kappa a_\varepsilon(x)|u_\varepsilon|^\alpha |v_\varepsilon|^\beta\big]dx \\
&= \int_{\Omega\cap B_1^c} \big[\lambda a_\varepsilon(x)|u_\varepsilon|^{2^*(s)} + \mu a_\varepsilon(x)|v_\varepsilon|^{2^*(s)} + 2^*(s)\kappa a_\varepsilon(x)|u_\varepsilon|^\alpha |v_\varepsilon|^\beta\big]dx.
\end{aligned}$$

Combined with the fact of that

$$\int_{\Omega\cap B_1}\left[\lambda a_\varepsilon(x)|u_\varepsilon|^{2^*(s)}+\mu a_\varepsilon(x)|v_\varepsilon|^{2^*(s)}+2^*(s)\kappa a_\varepsilon(x)|u_\varepsilon|^\alpha|v_\varepsilon|^\beta\right]dx$$
$$+\int_{\Omega\cap B_1^c}\left[\lambda a_\varepsilon(x)|u_\varepsilon|^{2^*(s)}+\mu a_\varepsilon(x)|v_\varepsilon|^{2^*(s)}+2^*(s)\kappa a_\varepsilon(x)|u_\varepsilon|^\alpha|v_\varepsilon|^\beta\right]dx=1,$$

we obtain that

$$\begin{aligned}&\int_{\Omega\cap B_1}\left[\lambda a_\varepsilon(x)|u_\varepsilon|^{2^*(s)}+\mu a_\varepsilon(x)|v_\varepsilon|^{2^*(s)}+2^*(s)\kappa a_\varepsilon(x)|u_\varepsilon|^\alpha|v_\varepsilon|^\beta\right]dx\\&=\int_{\Omega\cap B_1^c}\left[\lambda a_\varepsilon(x)|u_\varepsilon|^{2^*(s)}+\mu a_\varepsilon(x)|v_\varepsilon|^{2^*(s)}+2^*(s)\kappa a_\varepsilon(x)|u_\varepsilon|^\alpha|v_\varepsilon|^\beta\right]dx\\&=\frac{1}{2}.\end{aligned}\tag{7.55}$$

Hence, we have $\eta_r\equiv\frac{1}{2}$ for any $r>0$, a contradiction. □

Proof of Theorem 7.1: Let $(u_{\varepsilon_k},v_{\varepsilon_k})$ be a solution to (7.21) with $\varepsilon=\varepsilon_k$ given by Lemma 7.6 and $\varepsilon_k\to0$ as $k\to+\infty$. Up to a subsequence, we may assume that $u_{\varepsilon_k}\rightharpoonup u,v_{\varepsilon_k}\rightharpoonup v$ in $D_0^{1,2}(\Omega)$ and $u_\varepsilon\to u,v_\varepsilon\to v$ a.e. in Ω (see Lemma 7.7). Then if $0\not\in\bar{\Omega}$, by Rellich-Kondrachov compact theorem, it is easy to see that $u_\varepsilon\to u,v_{\varepsilon_k}\to v$ strongly in $D_0^{1,2}(\Omega)$. When Ω is a cone, by Corollary 7.2, we also obtain that $u_\varepsilon\to u,v_{\varepsilon_k}\to v$ strongly in $D_0^{1,2}(\Omega)$. By (iv) of Lemma 7.7, we obtain that (u,v) is an extremal of $S_{\alpha,\beta,\lambda,\mu}(\Omega)$, the proof is completed. □

Remark 7.3. *When Ω is a cone, let (u,v) be the extremal obtained as limit of $(u_{\varepsilon_k},v_{\varepsilon_k})$, the solution to* (7.21) *with $\varepsilon=\varepsilon_k$ given by Lemma 7.6. Then by Lemma 7.7, Corollary 7.2 and the formula* (7.55)*, we see that (u,v) satisfies*

$$\begin{aligned}&\int_{\Omega\cap B_1}\left[\lambda a_0(x)|u|^{2^*(s)}+\mu a_0(x)|v|^{2^*(s)}+2^*(s)\kappa a_0(x)|u|^\alpha|v|^\beta\right]dx\\&=\int_{\Omega\cap B_1^c}\left[\lambda a_0(x)|u|^{2^*(s)}+\mu a_0(x)|v|^{2^*(s)}+2^*(s)\kappa a_0(x)|u|^\alpha|v|^\beta\right]dx\\&=\frac{1}{2}.\end{aligned}$$

Such a property has been observed for the scalar equation.

7.3 Existence of positive ground state solutions

By (7.6), we always have $S_{\alpha,\beta,\lambda,\mu}(\Omega)\leq\left(\max\{\lambda,\mu\}\right)^{-\frac{2}{2^*(s)}}\mu_s(\Omega)$. When Ω is a cone and $s\in(0,2),\alpha>1,\beta>1,\alpha+\beta=2^*(s)$, by Theorem 7.1, $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ is always attained (although the extremals may be semi-trivial). Indeed, if $S_{\alpha,\beta,\lambda,\mu}(\Omega)=$

$\big(\max\{\lambda,\mu\}\big)^{-\frac{2}{2^*(s)}}\mu_s(\Omega)$, then $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ can be achieved by semi-trivial function. To see this, we just need to plug in the pairs $(U,0)$ or $(0,U)$, where U is an extremal function of $\mu_s(\Omega)$. But, under some special conditions, $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ can also be achieved by nontrivial function even $S_{\alpha,\beta,\lambda,\mu}(\Omega)=\big(\max\{\lambda,\mu\}\big)^{-\frac{2}{2^*(s)}}\mu_s(\Omega)$, see Theorem 7.4 below. However, if

$$S_{\alpha,\beta,\lambda,\mu}(\Omega)<\big(\max\{\lambda,\mu\}\big)^{-\frac{2}{2^*(s)}}\mu_s(\Omega), \tag{7.56}$$

then the extremal functions (hence the positive ground state of the system (7.1)) of $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ must be nontrivial. Therefore, next we need to search some sufficient conditions to ensure the above strict inequality (7.56). We obtain the following theorem on the existence, regularity and decay estimate.

Theorem 7.2. *Let Ω be a cone in $\mathbb{R}^N$(especially, $\Omega=\mathbb{R}^N$ or $\Omega=\mathbb{R}^N_+$) or Ω be an open domain but $0\notin\bar{\Omega}$. Assume $s\in(0,2),\kappa>0,\alpha>1,\beta>1,\alpha+\beta=2^*(s)$. Then system* (7.1) *possesses a positive ground state solution (ϕ,ψ) (i.e., $\phi>0,\psi>0$) provided that one of the following conditions holds:*

(a_1) $\lambda>\mu$ *and either* $1<\beta<2$ *or* $\begin{cases}\beta=2\\ \kappa>\frac{\lambda}{2^*(s)}\end{cases}$ *;*

(a_2) $\lambda=\mu$ *and either* $\min\{\alpha,\beta\}<2$ *or* $\begin{cases}\min\{\alpha,\beta\}=2,\\ \kappa>\frac{\lambda}{2^*(s)}\end{cases}$ *;*

(a_3) $\lambda<\mu$ *and either* $1<\alpha<2$ *or* $\begin{cases}\alpha=2\\ \kappa>\frac{\mu}{2^*(s)}\end{cases}$ *.*

Moreover, when Ω is a cone, we have the following regularity and decay properties:

(b_1) *if* $0<s<\frac{N+2}{N}$, $\phi,\psi\in C^2(\overline{\Omega})$*;*

(b_2) *if* $s=\frac{N+2}{N}$, $\phi,\psi\in C^{1,\gamma}(\Omega)$ *for all* $0<\gamma<1$*;*

(b_3) *if* $s>\frac{N+2}{N}$, $\phi,\psi\in C^{1,\gamma}(\Omega)$ *for all* $0<\gamma<\frac{N(2-s)}{N-2}$.

When Ω is a cone with $0\in\partial\Omega$ (e.g., $\Omega=\mathbb{R}^N_+$), then there exists a constant C such that

$$|\phi(x)|,|\psi(x)|\le C(1+|x|^{-(N-1)}),\qquad |\nabla\phi(x)|,|\nabla\psi(x)|\le C|x|^{-N}.$$

When $\Omega=\mathbb{R}^N$,

$$|\phi(x)|,|\psi(x)|\le C(1+|x|^{-N}),\qquad |\nabla\phi(x)|,|\nabla\psi(x)|\le C|x|^{-N-1}$$

In particular, if $\Omega=\mathbb{R}^N_+$, then $(\phi(x),\psi(x))$ is axially symmetric with respect to the x_N-axis, i.e.,

$$\big(\phi(x),\psi(x)\big)=\big(\phi(x',x_N),\psi(x',x_N)\big)=\big(\phi(|x'|,x_N),\psi(|x'|,x_N)\big).$$

Remark 7.4. *The conditions $(a_1)-(a_3)$ imposed in Theorem 7.2 are some sufficient conditions to ensure the inequality* (7.56) *(see Lemma 7.9 below). But they are not necessary conditions. For example, when $\lambda > \mu$ and $1 < \alpha < 2$, we can not exclude that $S_{\alpha,\beta,\lambda,\mu}(\Omega) < \lambda^{-\frac{2}{2^*(s)}}\mu_s(\Omega)$, i.e.,* (7.56) *may be true.*

Define the functional

$$\begin{aligned}\Phi(u,v) =&\frac{1}{2}\int_\Omega \big(|\nabla u|^2+|\nabla v|^2\big)dx\\ &-\frac{1}{2^*(s)}\int_\Omega\frac{1}{|x|^s}\big[\lambda|u|^{2^*(s)}+\mu|v|^{2^*(s)}+2^*(s)\kappa|u|^\alpha|v|^\beta\big]dx\end{aligned}\tag{7.57}$$

and the corresponding Nehari manifold

$$\mathcal{N}:=\{(u,v)\in\mathscr{D}\backslash\{0,0\}:\ J(u,v)=0\}\tag{7.58}$$

where

$$J(u,v)\tag{7.59}$$

$$\begin{aligned}&:=\langle\Phi'(u,v),(u,v)\rangle\\ &=\int_\Omega\big(|\nabla u|^2+|\nabla v|^2\big)dx-\int_\Omega\frac{1}{|x|^s}\big[\lambda|u|^{2^*(s)}+\mu|v|^{2^*(s)}+2^*(s)\kappa|u|^\alpha|v|^\beta\big]dx.\end{aligned}\tag{7.60}$$

By Lemma 4.1, $\mathcal{N}$ is well defined. Define

$$c_0:=\inf_{(u,v)\in\mathcal{N}}\Phi(u,v),\tag{7.61}$$

then basing on the results of Section 4 and Section 6, we see that

$$0<c_0\leq[\frac{1}{2}-\frac{1}{2^*(s)}]\big[\mu_s(\Omega)\big]^{\frac{2^*(s)}{2^*(s)-2}}\big(\max\{\lambda,\mu\}\big)^{-\frac{2}{2^*(s)-2}}.\tag{7.62}$$

Moreover, we have the following result.

Lemma 7.8. *Let Ω be a cone of $\mathbb{R}^N$ or Ω be an open domain but $0\notin\bar{\Omega}$. Assume that $\kappa>0, s\in(0,2), \alpha>1, \beta>1, \alpha+\beta=2^*(s)$ and let c_0 be defined by* (7.61)*, then we have*

$$c_0<[\frac{1}{2}-\frac{1}{2^*(s)}]\big(\mu_s(\Omega)\big)^{\frac{2^*(s)}{2^*(s)-2}}\big(\max\{\lambda,\mu\}\big)^{-\frac{2}{2^*(s)-2}}\tag{7.63}$$

if and only if

$$S_{\alpha,\beta,\lambda,\mu}(\Omega)<\big(\max\{\lambda,\mu\}\big)^{-\frac{2}{2^*(s)}}\mu_s(\Omega).\tag{7.64}$$

Proof. A direct computation shows that

$$c_0=[\frac{1}{2}-\frac{1}{2^*(s)}]\big(S_{\alpha,\beta,\lambda,\mu}(\Omega)\big)^{\frac{2^*(s)}{2^*(s)-2}}.\tag{7.65}$$

□

Then combining with the conclusions of Section 6, we have the following result:

Lemma 7.9. *Let Ω be a cone in $\mathbb{R}^N$(especially, $\Omega = \mathbb{R}^N$ and $\Omega = \mathbb{R}^N_+$) or Ω be an open domain but $0 \notin \bar{\Omega}$. Suppose $s \in (0,2), \kappa > 0, \alpha > 1, \beta > 1, \alpha + \beta = 2^*(s)$. Then $S_{\alpha,\beta,\lambda,\mu}(\Omega) < \big(\max\{\lambda,\mu\}\big)^{-\frac{2}{2^*(s)}}\mu_s(\Omega)$ if one of the following holds:*

(i) $\lambda > \mu$ *and either* $1 < \beta < 2$ *or* $\begin{cases}\beta = 2\\ \kappa > \frac{\lambda}{2^*(s)}\end{cases}$;

(ii) $\lambda = \mu$ *and either* $\min\{\alpha,\beta\} < 2$ *or* $\begin{cases}\min\{\alpha,\beta\} = 2,\\ \kappa > \frac{\lambda}{2^*(s)}\end{cases}$;

(iii) $\lambda < \mu$ *and either* $1 < \alpha < 2$ *or* $\begin{cases}\alpha = 2\\ \kappa > \frac{\mu}{2^*(s)}\end{cases}$.

Proof. It follows by Corollary 6.3, Corollary 6.4, Lemma 6.5 and Lemma 7.8. □

Proof of Theorem 7.2: Under the assumptions of Theorem 7.2, firstly by Theorem 7.1, $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ is attained by some nonnegative pair (u,v) such that $(u,v) \neq (0,0)$. On the other hand, by Lemma 7.9, we have

$$S_{\alpha,\beta,\lambda,\mu}(\Omega) < \big(\max\{\lambda,\mu\}\big)^{-\frac{2}{2^*(s)}}\mu_s(\Omega).$$

Hence, we see that $u \neq 0, v \neq 0$. Hence, by Proposition 7.1,

$$(\phi,\psi) := \Big(\big(S_{\alpha,\beta,\lambda,\mu}(\Omega)\big)^{\frac{1}{2^*(s)-2}}u, \big(S_{\alpha,\beta,\lambda,\mu}(\Omega)\big)^{\frac{1}{2^*(s)-2}}v\Big)$$

is a ground state solution of system (7.1). Then by the strong maximum principle, it is easy to see that $\phi > 0, , \psi > 0$ in Ω. We note that the arguments in Proposition 3.1 and Proposition 3.2 are valid for general cone. Combining with Proposition 3.3, we complete the proof. □

7.4 Uniqueness and Nonexistence of positive ground state solutions

In the previous subsection, in Theorem 7.2, we have established the existence of the positive ground state solution to the system (7.1). Now, in the current subsection, we obtain the uniqueness of the positive ground state solution to the system (7.1). Define

$$G(u,v) := \frac{\int_\Omega \big(|\nabla u|^2 + |\nabla v|^2\big)dx}{\Big(\int_\Omega \big(\lambda\frac{|u|^{2^*(s)}}{|x|^s} + \mu\frac{|v|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa\frac{|u|^\alpha|v|^\beta}{|x|^s}\big)dx\Big)^{\frac{2}{2^*(s)}}}, \quad (u,v) \neq (0,0) \tag{7.66}$$

then we have

$$S_{\alpha,\beta,\lambda,\mu}(\Omega) = \inf_{(u,v)\in\mathscr{D}\setminus\{(0,0)\}} G(u,v). \tag{7.67}$$

For any $u \neq 0, v \neq 0$ and $t \geq 0$, we have

$$G(u,tv) = \frac{\int_\Omega \big(|\nabla u|^2 + |\nabla v|^2 t^2\big)dx}{\Big(\int_\Omega \big(\lambda \frac{|u|^{2^*(s)}}{|x|^s} + \mu \frac{|v|^{2^*(s)}}{|x|^s} t^{2^*(s)} + 2^*(s)\kappa \frac{|u|^\alpha |v|^\beta}{|x|^s} t^\beta\big)dx\Big)^{\frac{2}{2^*(s)}}}. \quad (7.68)$$

Hence,

$$\begin{aligned}
G(u,tu) =& \frac{\int_\Omega \big(|\nabla u|^2 + |\nabla u|^2 t^2\big)dx}{\Big(\int_\Omega \big(\lambda \frac{|u|^{2^*(s)}}{|x|^s} + \mu \frac{|u|^{2^*(s)}}{|x|^s} t^{2^*(s)} + 2^*(s)\kappa \frac{|u|^\alpha |u|^\beta}{|x|^s} t^\beta\big)dx\Big)^{\frac{2}{2^*(s)}}} \\
=& \frac{1+t^2}{\big[\lambda + \mu t^{2^*(s)} + 2^*(s)\kappa t^\beta\big]^{\frac{2}{2^*(s)}}} \frac{\int_\Omega |\nabla u|^2 dx}{\Big(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx\Big)^{\frac{2}{2^*(s)}}} \\
:=& g(t) \frac{\int_\Omega |\nabla u|^2 dx}{\Big(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx\Big)^{\frac{2}{2^*(s)}}}. \qquad (7.69)
\end{aligned}$$

We define $g(+\infty) = \lim_{t\to+\infty} g(t) = \mu^{-\frac{2}{2^*(s)}}$, then we see that

$$G(0,v) = \lim_{t\to+\infty} G(v,tv) = g(+\infty) \frac{\int_\Omega |\nabla v|^2 dx}{\Big(\int_\Omega \frac{|v|^{2^*(s)}}{|x|^s} dx\Big)^{\frac{2}{2^*(s)}}}.$$

Hence, we have

$$\begin{aligned}
S_{\alpha,\beta,\lambda,\mu}(\Omega) =& \inf_{(u,v)\in\mathscr{D}\setminus\{(0,0)\}} G(u,v) \\
\leq& \inf_{u\in D_0^{1,2}(\Omega)} \inf_{t\in[0,+\infty)} G(u,tu) \\
=& \inf_{t\in[0,+\infty)} g(t) \inf_{u\in D_0^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\Big(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx\Big)^{\frac{2}{2^*(s)}}} \\
=& \inf_{t\in[0,+\infty)} g(t)\mu_s(\Omega). \qquad (7.70)
\end{aligned}$$

Moreover, we can obtain the follow precise result:

Lemma 7.10. $S_{\alpha,\beta,\lambda,\mu}(\Omega) = \inf_{t\in[0,+\infty)} g(t)\mu_s(\Omega)$, *where*

$$g(t) := \frac{1+t^2}{\big[\lambda + \mu t^{2^*(s)} + 2^*(s)\kappa t^\beta\big]^{\frac{2}{2^*(s)}}}. \quad (7.71)$$

Proof. By (7.70), we only need to prove the reverse inequality. Now, let $\{(u_n, v_n)\}$ be a minimizing sequence of $S_{\alpha,\beta,\lambda,\mu}(\Omega)$. Since $G(u,v) = G(tu,tv)$ for all $t > 0$, without loss of generality, we may assume that

$$\int_\Omega \big(\frac{|u_n|^{2^*(s)}}{|x|^s} + \frac{|v_n|^{2^*(s)}}{|x|^s}\big) \equiv 1,$$

and

$$G(u_n, v_n) = S_{\alpha,\beta,\lambda,\mu}(\Omega) + o(1).$$

Case 1: $\liminf\limits_{n\to+\infty} \int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx = 0$. Since $\{v_n\}$ is bounded in $L^{2^*(s)}(\Omega, \frac{dx}{|x|^s})$, by Hölder inequality, up to a subseqeunce, we see that

$$\int_\Omega \big(\lambda \frac{|u_n|^{2^*(s)}}{|x|^s} + \mu \frac{|v_n|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa \frac{|u_n|^\alpha |v_n|^\beta}{|x|^s}\big) dx = \mu \int_\Omega \frac{|v_n|^{2^*(s)}}{|x|^s} dx + o(1) = \mu + o(1).$$

Hence,

$$\lim_{n\to\infty} G(u, v_n) \geq \lim_{n\to\infty} G(0, v_n).$$

We see that $(0, v_n)$ is also a minimizing sequence of $S_{\alpha,\beta,\lambda,\mu}(\Omega)$. Hence, it is easy to see that

$$\begin{aligned} S_{\alpha,\beta,\lambda,\mu}(\Omega) =& \mu^{-\frac{2}{2^*(s)}} \mu_s(\Omega) \\ =& g(+\infty)\mu_s(\Omega) \\ \geq& \inf_{t\in(0,+\infty)} g(t)\mu_s(\Omega). \end{aligned} \tag{7.72}$$

Case 2: $\liminf\limits_{n\to+\infty} \int_\Omega \frac{|v_n|^{2^*(s)}}{|x|^s} dx = 0$. Similarly to Case 1, we can obtain that

$$\begin{aligned} S_{\alpha,\beta,\lambda,\mu}(\Omega) =& \lambda^{-\frac{2}{2^*(s)}} \mu_s(\Omega) \\ =& g(0)\mu_s(\Omega) \\ \geq& \inf_{t\in[0,+\infty)} g(t)\mu_s(\Omega). \end{aligned} \tag{7.73}$$

Case 3: Up to a subseqeuce if necessary, we may assume that $\lim\limits_{n\to+\infty} \int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx = \delta > 0$ and $\lim\limits_{n\to+\infty} \int_\Omega \frac{|v_n|^{2^*(s)}}{|x|^s} dx = 1 - \delta > 0$. Let $t_n > 0$ such that

$$\int_\Omega \frac{|v_n|^{2^*(s)}}{|x|^s} dx = \int_\Omega \frac{|t_n u_n|^{2^*(s)}}{|x|^s} dx,$$

then we see that $\{t_n\}$ is bounded and away from 0. Up to a subsequence, we may assume that $t_n \to t_0 = \left(\frac{\delta}{1-\delta}\right)^{\frac{1}{2^*(s)}}$. Now let $w_n = \frac{1}{t_n} v_n$, then we have

$$\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx = \int_\Omega \frac{|w_n|^{2^*(s)}}{|x|^s} dx \tag{7.74}$$

and by Young's inequality, we have

$$\begin{aligned} \int_\Omega \frac{|u_n|^\alpha |w_n|^\beta}{|x|^s} dx \leq& \frac{\alpha}{2^*(s)} \int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx + \frac{\beta}{2^*(s)} \int_\Omega \frac{|w_n|^{2^*(s)}}{|x|^s} dx \\ =& \int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx = \int_\Omega \frac{|w_n|^{2^*(s)}}{|x|^s} dx. \end{aligned} \tag{7.75}$$

Hence,

$$
\begin{aligned}
G(u_n,v_n) =&G(u_n,t_nw_n)\\
=&\frac{\int_\Omega |\nabla u_n|^2dx}{\Big(\int_\Omega\big(\lambda\frac{|u_n|^{2^*(s)}}{|x|^s}+\mu t_n^{2^*(s)}\frac{|w_n|^{2^*(s)}}{|x|^s}+2^*(s)\kappa t_n^\beta\frac{|u_n|^\alpha|w_n|^\beta}{|x|^s}\big)dx\Big)^{\frac{2}{2^*(s)}}}\\
&+\frac{\int_\Omega t_n^2|\nabla w_n|^2dx}{\Big(\int_\Omega\big(\lambda\frac{|u_n|^{2^*(s)}}{|x|^s}+\mu t_n^{2^*(s)}\frac{|w_n|^{2^*(s)}}{|x|^s}+2^*(s)\kappa t_n^\beta\frac{|u_n|^\alpha|w_n|^\beta}{|x|^s}\big)dx\Big)^{\frac{2}{2^*(s)}}}\\
\geq&\frac{1}{\Big[\lambda+\mu t_n^{2^*(s)}+2^*(s)\kappa t_n^\beta\Big]^{\frac{2}{2^*(s)}}}\frac{\int_\Omega |\nabla u_n|^2dx}{\Big(\int_\Omega\frac{|u_n|^{2^*(s)}}{|x|^s}dx\Big)^{\frac{2}{2^*(s)}}}\\
&+\frac{t_n^2}{\Big[\lambda+\mu t_n^{2^*(s)}+2^*(s)\kappa t_n^\beta\Big]^{\frac{2}{2^*(s)}}}\frac{\int_\Omega |\nabla w_n|^2dx}{\Big(\int_\Omega\frac{|w_n|^{2^*(s)}}{|x|^s}dx\Big)^{\frac{2}{2^*(s)}}}\\
\geq&\frac{1}{\Big[\lambda+\mu t_n^{2^*(s)}+2^*(s)\kappa t_n^\beta\Big]^{\frac{2}{2^*(s)}}}\mu_s(\Omega)\\
&+\frac{t_n^2}{\Big[\lambda+\mu t_n^{2^*(s)}+2^*(s)\kappa t_n^\beta\Big]^{\frac{2}{2^*(s)}}}\mu_s(\Omega)\\
=&g(t_n)\mu_s(\Omega).
\end{aligned}\tag{7.76}
$$

Let $n\to+\infty$, we obtain that

$$S_{\alpha,\beta,\lambda,\mu}(\Omega)\geq g(t_0)\mu_s(\Omega)\geq\inf_{t\in(0,+\infty)}g(t)\mu_s(\Omega).$$

Thereby $S_{\alpha,\beta,\lambda,\mu}(\Omega)=\inf_{t\in(0,+\infty)}g(t)\mu_s(\Omega)$ is proved. $\square$

Basing on Lemma 7.10, we can propose the "uniqueness" type result as following:

Theorem 7.3. *Let Ω either be a cone in $\mathbb{R}^N$ (in particular, $\Omega=\mathbb{R}^N$ and $\Omega=\mathbb{R}^N_+$) or Ω be an open domain but $0\notin\bar\Omega$. Assume $s\in(0,2),\kappa>0,\alpha>1,\beta>1,\alpha+\beta=2^*(s)$. Let (ϕ,ψ) be a positive ground state solution to problem* (7.1)*, then*

$$\phi=C(t_0)U,\quad \psi=t_0C(t_0)U,$$

where U is the ground state solution of

$$\begin{cases}-\Delta u=\mu_s(\Omega)\frac{u^{2^*(s)-1}}{|x|^s}\ \text{in } \Omega,\\ u=0\ \text{ on } \partial\Omega,\end{cases}\tag{7.77}$$

while $t_0>0$ satisfies that

$$g(t_0)=\inf_{t\in(0,+\infty)}g(t)\tag{7.78}$$

and

$$C(t_0) := [S_{\alpha,\beta,\lambda,\mu}(\Omega)]^{\frac{1}{2^*(s)-2}} \left(\frac{1}{\lambda + \mu t_0^{2^*(s)} + 2^*(s)\kappa t_0^{\beta}} \right)^{\frac{1}{2^*(s)}}, \tag{7.79}$$

where $g(t)$ *is defined in (7.71).*

Proof. By the processes of Case 3 in the proof of Lemma 7.10, if $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ is attained by some nontrivial function (u,v), i.e., $u \neq 0, v \neq 0$, then there exists some $t_0 > 0$ such that $v = t_0 u$, where u is a minimizer of $\mu_s(\Omega)$ and t_0 satisfies $g(t_0) = \inf\limits_{t\in(0,+\infty)} g(t)$.

Now assume that $u = CU, v = t_0 CU$, then a direct computation shows that

$$\int_\Omega \frac{1}{|x|^s} \big[\lambda |u|^{2^*(s)} + \mu |v|^{2^*(s)} + 2^*(s)\kappa |u|^\alpha |v|^\beta \big] dx = 1$$

if and only if

$$C = \left(\frac{1}{\lambda + \mu t_0^{2^*(s)} + 2^*(s)\kappa t_0^{\beta}} \right)^{\frac{1}{2^*(s)}}.$$

Finally, we see that $\phi = [S_{\alpha,\beta,\lambda,\mu}(\Omega)]^{\frac{1}{2^*(s)-2}} u, \psi = [S_{\alpha,\beta,\lambda,\mu}(\Omega)]^{\frac{1}{2^*(s)-2}} v$, we complete the proof. □

Remark 7.5. *Under the assumption that*

$$S_{\alpha,\beta,\lambda,\mu}(\Omega) < \big(\max\{\lambda,\mu\}\big)^{-\frac{2}{2^*(s)}} \mu_s(\Omega),$$

we have seen that the problem (7.1) *possesses a positive ground state solution. But the converse usually is not true. Next, we construct an example where* $S_{\alpha,\beta,\lambda,\mu}(\Omega) = \big(\max\{\lambda,\mu\}\big)^{-\frac{2}{2^*(s)}} \mu_s(\Omega)$ *but the system* (7.1) *still has (multiple) positive ground state solutions.*

Theorem 7.4. *Let* Ω *be a cone in* $\mathbb{R}^3$ *or* Ω *be an open domain but* $0 \notin \bar{\Omega}$*. Assume the following conditions hold:*

(a) $0 < s < 2$,

(b) *either* $\alpha > 2$ *or* $\begin{cases} \alpha = 2 \\ \mu \geq 2\kappa \end{cases}$,

(c) *either* $\beta > 2$ *or* $\begin{cases} \beta = 2 \\ \lambda \geq 2\kappa \end{cases}$,

(d) $\alpha + \beta = 2^*(s)$.

Then we have one of the following conclusion:

(1) If $s = 1, \alpha = \beta = 2, \lambda = \mu = 2\kappa > 0$, then the set of all extremal functions of $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ is given by

$$\mathcal{A} := \Big\{(t_1 U, t_2 U) :\ t_1 \geq 0, t_2 \geq 0, (t_1, t_2) \neq (0,0) \text{ and}$$
$$U \text{ is an extremal of } \mu_s(\Omega)\Big\}. \tag{7.80}$$

(2) Except for the item (1) above, $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ has no nontrivial extremal function.

Remark 7.6. *Under the hypotheses (a)-(d), the dimension of the space $\mathbb{R}^N$ has to be three. Therefore, we can only establish the above theorem in $\mathbb{R}^3$.*

Proof. We proceed by contradiction. Assume that $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ has a nontrivial extremal (u, v), then by Lemma 7.10 (see case 3 of the proof), we see that there exists some $t_0 > 0$ such that $v = t_0 u$ and u is an extremal of $\mu_s(\Omega)$. Moreover, t_0 attains the minimum of $g(t)$, where $g(t)$ is introduced in (7.71). By conditions (b) and (c), we see that $g''(0) \geq 0$ and $g'(t) < 0$ for t large enough. Hence, $\{t > 0 :\ g'(t) = 0\}$ has at least 3 solutions $\{t_1, t_2, t_3\}$ such that $0 < t_1 < t_2 = t_0 < t_3 < \infty$. A direct computation shows that

$$\begin{aligned} g'(t) =& \frac{-2\mu t^{2^*(s)-1} + 2\kappa\alpha t^{\beta+1} - 2\kappa\beta t^{\beta-1} + 2\lambda t}{\big[\lambda + \mu t^{2^*(s)} + 2^*(s)\kappa t^{\beta}\big]^{\frac{2}{2^*(s)}+1}} \\ =& \frac{-2t}{\big[\lambda + \mu t^{2^*(s)} + 2^*(s)\kappa t^{\beta}\big]^{\frac{2}{2^*(s)}+1}} \ \big(\mu t^{2^*(s)-2} - \kappa\alpha t^{\beta} + \kappa\beta t^{\beta-2} - \lambda\big). \end{aligned} \tag{7.81}$$

Define

$$h(t) := \mu t^{2^*(s)-2} - \kappa\alpha t^{\beta} + \kappa\beta t^{\beta-2} - \lambda, \tag{7.82}$$

then we obtain that $\{t > 0 :\ h(t) = 0\}$ has at least 3 solutions $\{t_1, t_2, t_3\}$ such that $0 < t_1 < t_2 = t_0 < t_3 < \infty$.

Case 1:$\beta = 2$ **and** $2\kappa - \lambda \leq 0$**.** For this case, $h(t) = \mu t^{2^*(s)-2} - \kappa\alpha t^2 + 2\kappa - \lambda$. By Rolle's mean value theorem, $\{t > 0 :\ h'(t) = 0\}$ has at least two solutions $\tilde{t}_1, \tilde{t}_2$ such that

$$t_1 < \tilde{t}_1 < t_2 = t_0 < \tilde{t}_2 < t_3.$$

Note that

$$\{t > 0 :\ h'(t) = 0\} = \{t > 0 :\ \mu[2^*(s) - 2]t^{2^*(s)-4} - 2\kappa\alpha = 0\}.$$

In particular, the set $\{t > 0 :\ \mu[2^*(s) - 2]t^{2^*(s)-4} - 2\kappa\alpha = 0\}$ has a unique solution if $2^*(s) \neq 4$, a contradiction. Hence, $2^*(s) = 4$ and $\mu = 2\kappa$. Recalling that $2^*(s) = \frac{2(N-s)}{N-2}, s \in (0, 2)$, we obtain that $s = 1, \alpha = 2^*(s) - \beta = 2$. Then

$$g(t_0) = \frac{1 + t_0^2}{[\lambda + 2\kappa t_0^4 + 4\kappa t_0^2]^{\frac{1}{2}}} = \frac{1}{\sqrt{\lambda}}, \tag{7.83}$$

which implies that

$$\lambda = \mu = 2\kappa.$$

It follows that $g(t) \equiv \frac{1}{\sqrt{\lambda}}$. Hence, when $N = 3, s = 1, \alpha = \beta = 2, \lambda = \mu = 2\kappa$, the extremals of $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ are given by (7.80). In particular,

$$\{(\phi, \psi) = \sqrt{\frac{\mu_s(\Omega)}{2\kappa(1+t^2)}}(U, tU) : t > 0\}$$

are all the ground state solutions of

$$\begin{cases} -\Delta u - 2\kappa\frac{|u|^2 u}{|x|} = 2\kappa\frac{uv^2}{|x|} \text{ in } \Omega, \\ -\Delta v - 2\kappa\frac{|v|^2 v}{|x|} = 2\kappa\frac{u^2 v}{|x|} \text{ in } \Omega, \\ \kappa > 0, u, v \in D_0^{1,2}(\Omega), \end{cases} \tag{7.84}$$

where U is the ground state solution of (7.77).

Case 2: $\beta > 2$**.** For this case, $h(t) = \mu t^{2^*(s)-2} - \kappa\alpha t^\beta + \kappa\beta t^{\beta-2} - \lambda$, similarly we see that the equation $h(t) = 0$ ($t > 0$) has at least three roots $t_1 < t_2 = t_0 < t_3$. It follows that $\{h'(t), t > 0\}$ has at least two roots $\tilde{t}_1$ and $\tilde{t}_2$, which implies that $p(t) = 0$ $(t > 0)$ has at least two solutions. Where $p(t)$ is defined by

$$p(t) := \mu[2^*(s) - 2]t^{2^*(s)-\beta} - \kappa\alpha\beta t^2 + \kappa\beta(\beta - 2). \tag{7.85}$$

A direct computation shows that $p''(t) > 0$ when $\alpha > 2$. Hence $p(t) = 0$ $(t > 0)$ could not have more than one solution, a contradiction. If $\beta > 2, \alpha = 2$, we have $p(t) = \big[\mu[2^*(s) - 2] - \kappa\alpha\beta\big]t^2 + \kappa\beta(\beta - 2)$, which also has at most one positive root, a contradiction too.

We note that for the case of $\mu > \lambda$, we will take

$$\tilde{g}(t) := g(\frac{1}{t}) = \frac{1 + t^2}{\big[\mu + \lambda t^{2^*(s)} + 2^*(s)\kappa t^\alpha\big]^{\frac{2}{2^*(s)}}}$$

and the arguments above can repeated (β is replaced by α now). We complete the proof. □

Corollary 7.3. *Under the assumptions of Theorem 7.4, there must hold*

$$S_{\alpha,\beta,\lambda,\mu}(\Omega) = \big(\max\{\lambda, \mu\}\big)^{-\frac{2}{2^*(s)}}\mu_s(\Omega).$$

Proof. For the special case $N = 3, s = 1, \alpha = \beta = 2, \lambda = \mu = 2\kappa$, a direct computation can deduce it. For the other cases, if $S_{\alpha,\beta,\lambda,\mu}(\Omega) \neq \big(\max\{\lambda, \mu\}\big)^{-\frac{2}{2^*(s)}}\mu_s(\Omega)$, then there must hold that $S_{\alpha,\beta,\lambda,\mu}(\Omega) < \big(\max\{\lambda, \mu\}\big)^{-\frac{2}{2^*(s)}}\mu_s(\Omega)$. By Theorem 7.1 and Lemma 7.8, $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ can be achieved by some nontrivial extremal (u, v), a contradiction to Theorem 7.4. □

7.5 Further results about cones

Assume that $0 < s < 2, \alpha > 1, \beta > 1, \alpha + \beta = 2^*(s)$. Based on the results of Theorem 7.1, we see that when Ω is a cone, $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ is always achieved. In this subsection, we always assume that Ω is a cone. We shall investigate more properties about $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ in terms of Ω. Let us begin with a remark.

Remark 7.7. *Assume that Ω_1, Ω_2 are domains of $\mathbb{R}^N$ and $\Omega_1 \subseteq \Omega_2$, then it is easy to see that $D_0^{1,2}(\Omega_1) \subseteq D_0^{1,2}(\Omega_2)$. Then by the definition of $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ (see the formula (7.2)), we see that $S_{\alpha,\beta,\lambda,\mu}(\Omega_1) \geq S_{\alpha,\beta,\lambda,\mu}(\Omega_2)$.*

Lemma 7.11. *Let $\{\Omega_n\}$ be a sequence of cones.*

(i) Assume $\{\Omega_n\}$ is an increasing sequence, i.e., $\Omega_n \subseteq \Omega_{n+1}$, then

$$\lim_{n\to\infty} S_{\alpha,\beta,\lambda,\mu}(\Omega_n) = S_{\alpha,\beta,\lambda,\mu}(\lim_{n\to\infty} \Omega_n) = S_{\alpha,\beta,\lambda,\mu}(\Omega),$$

where

$$\Omega = \lim_{n\to\infty} \Omega_n = \bigcup_{n=1}^{\infty} \Omega_n.$$

(ii) Assume $\{\Omega_n\}$ is a decreasing sequence, i.e., $\Omega_n \supseteq \Omega_{n+1}$, then

$$\lim_{n\to\infty} S_{\alpha,\beta,\lambda,\mu}(\Omega_n) = S_{\alpha,\beta,\lambda,\mu}(\lim_{n\to\infty} \Omega_n) = S_{\alpha,\beta,\lambda,\mu}(\Omega),$$

where

$$\Omega = \bigcap_{n=1}^{\infty} \Omega_n,$$

and we denote that $S_{\alpha,\beta,\lambda,\mu}(\Omega) = +\infty$ if $meas(\Omega) = 0$.

Proof. (i) By Remark 7.7, we see that $\{S_{\alpha,\beta,\lambda,\mu}(\Omega_n)\}$ is a decreasing nonnegative sequence. Hence $\lim_{n\to\infty} S_{\alpha,\beta,\lambda,\mu}(\Omega_n)$ exists. Also by Remark 7.7 and $\Omega = \bigcup_{i=1}^{\infty} \Omega_i \supseteq \Omega_n, \forall\, n$, we have

$$S_{\alpha,\beta,\lambda,\mu}(\Omega) \leq \lim_{n\to\infty} S_{\alpha,\beta,\lambda,\mu}(\Omega_n). \tag{7.86}$$

On the other hand, for any $\varepsilon > 0$, $\exists\ (u_\varepsilon, v_\varepsilon) \in C_c^\infty(\Omega) \times C_c^\infty(\Omega)$ such that

$$\int_\Omega [|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2] dx < S_{\alpha,\beta,\lambda,\mu}(\Omega) + \varepsilon \tag{7.87}$$

and

$$\int_\Omega \big(\lambda \frac{|u_\varepsilon|^{2^*(s)}}{|x|^s} + \mu \frac{|v_\varepsilon|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa \frac{|u_\varepsilon|^\alpha |v_\varepsilon|^\beta}{|x|^s}\big) dx = 1. \tag{7.88}$$

Then there exists some N_0 large enough such that

$$u_\varepsilon, v_\varepsilon \in C_c^\infty(\Omega_n) \text{ for all } n \geq N_0. \tag{7.89}$$

Hence, by the definition of $S_{\alpha,\beta,\lambda,\mu}(\Omega_n)$ again, we have

$$S_{\alpha,\beta,\lambda,\mu}(\Omega_n) < S_{\alpha,\beta,\lambda,\mu}(\Omega) + \varepsilon \text{ for all } n \geq N_0. \tag{7.90}$$

Let n go to infinity, we have

$$\lim_{n\to\infty} S_{\alpha,\beta,\lambda,\mu}(\Omega_n) \le S_{\alpha,\beta,\lambda,\mu}(\Omega) + \varepsilon. \tag{7.91}$$

By the arbitrariness of ε, we have

$$\lim_{n\to\infty} S_{\alpha,\beta,\lambda,\mu}(\Omega_n) \le S_{\alpha,\beta,\lambda,\mu}(\Omega). \tag{7.92}$$

Now, (7.86) and (7.92) say that

$$\lim_{n\to\infty} S_{\alpha,\beta,\lambda,\mu}(\Omega_n) = S_{\alpha,\beta,\lambda,\mu}(\lim_{n\to\infty}\Omega_n) = S_{\alpha,\beta,\lambda,\mu}(\Omega). \tag{7.93}$$

(ii) By Remark 7.7, we see that $\{S_{\alpha,\beta,\lambda,\mu}(\Omega_n)\}$ is an increasing sequence. Let us denote

$$\bar{S} := \lim_{n\to\infty} S_{\alpha,\beta,\lambda,\mu}(\Omega_n).$$

For any n, let (u_n, v_n) be the extremal function to $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ by Theorem 7.1. We can extend u_n and v_n by 0 out side Ω_n. By Remark 7.3, we have $\int_{\Omega_1}\big(\lambda\frac{|u_n|^{2^*(s)}}{|x|^s} + \mu\frac{|v_n|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa\frac{|u_n|^\alpha|v_n|^\beta}{|x|^s}\big)dx \equiv 1$ and

$$\begin{aligned}
&\int_{\Omega_1\cap B_1}\big(\lambda\frac{|u_n|^{2^*(s)}}{|x|^s} + \mu\frac{|v_n|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa\frac{|u_n|^\alpha|v_n|^\beta}{|x|^s}\big)dx\\
=&\int_{\Omega_1\setminus B_1}\big(\lambda\frac{|u_n|^{2^*(s)}}{|x|^s} + \mu\frac{|v_n|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa\frac{|u_n|^\alpha|v_n|^\beta}{|x|^s}\big)dx\\
=&\frac{1}{2}.
\end{aligned} \tag{7.94}$$

Case 1–$meas(\Omega)=0$: In this case, we shall prove that $\bar{S}=\infty$. Now,we proceed by contradiction. If $\bar{S}<\infty$, $\{u_n\},\{v_n\}$ are bounded sequences in $D_0^{1,2}(\Omega_1)$. Then up to a subsequence, we may assume that $u_n \rightharpoonup u, v_n \rightharpoonup v$ in $D_0^{1,2}(\Omega_1)$ and $u_n \to u, v_n \to v$ a.e. in Ω_1. Since $meas(\bigcap_{n=1}^\infty \Omega_n)=0$, we get $u=0, v=0$. On the other hand, by applying the same argument as Corollary 7.2, we can obtain that $u_n \to u, v_n \to v$ in $L^{2^*(s)}(\Omega_1, \frac{dx}{|x|^s})$. Then we have

$$\int_{\mathbb{R}^N}\big(\lambda\frac{|u|^{2^*(s)}}{|x|^s} + \mu\frac{|v|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa\frac{|u|^\alpha|v|^\beta}{|x|^s}\big)dx = 1, \tag{7.95}$$

a contradiction. Hence $\bar{S}=\infty$.

Case 2–Ω is a cone: In this case, by Theorem 7.1, $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ is well defined and can be achieved. Notice that for any n, we have $\Omega\subseteq\Omega_n$, by Remark 7.7 again, we have $S_{\alpha,\beta,\lambda,\mu}(\Omega_n)\le S_{\alpha,\beta,\lambda,\mu}(\Omega)$. Hence

$$\bar{S} \le S_{\alpha,\beta,\lambda,\mu}(\Omega). \tag{7.96}$$

Thus, $\{u_n\}, \{v_n\}$ are bounded in $D_0^{1,2}(\Omega_1)$ for this case. Arguing as before, it is easy to see the weak limit $u \neq 0, v \neq 0$ and

$$0 < \int_\Omega \big(\lambda \frac{|u|^{2^*(s)}}{|x|^s} + \mu \frac{|v|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa \frac{|u|^\alpha |v|^\beta}{|x|^s}\big) dx \leq 1. \tag{7.97}$$

We claim that (u, v) weakly solves

$$\begin{cases} -\Delta u = \bar{S}\Big(\lambda \frac{1}{|x|^s}|u|^{2^*(s)-2}u + \kappa\alpha \frac{1}{|x|^s}|u|^{\alpha-2}u|v|^\beta\Big) & \text{in } \Omega, \\ -\Delta v = \bar{S}\Big(\mu \frac{1}{|x|^s}|v|^{2^*(s)-2}v + \kappa\beta \frac{1}{|x|^s}|u|^{\alpha}|v|^{\beta-2}v\Big) & \text{in } \Omega, \\ u \geq 0, v \geq 0, (u, v) \in \mathscr{D} := D_0^{1,2}(\Omega) \times D_0^{1,2}(\Omega), \end{cases} \tag{7.98}$$

Since $C_c^\infty(\Omega)$ is dense in $D_0^{1,2}(\Omega)$, we only need to prove that

$$\begin{aligned} &\int_\Omega \nabla u \cdot \nabla \phi + \nabla v \cdot \nabla \psi \\ =&\bar{S} \int_\Omega \big(\lambda \frac{|u|^{2^*(s)-1}\phi}{|x|^s} + \mu \frac{|v|^{2^*(s)-1}\psi}{|x|^s} + \kappa\alpha \frac{|u|^{\alpha-2}u\phi|v|^\beta}{|x|^s} + \kappa\beta \frac{|u|^\alpha |v|^{\beta-2}v\psi}{|x|^s}\big) dx \\ &\text{for all } (\phi, \psi) \in C_c^\infty(\Omega) \times C_c^\infty(\Omega). \end{aligned} \tag{7.99}$$

Now, let $(\phi, \psi) \in C_c^\infty(\Omega) \times C_c^\infty(\Omega)$ be fixed. Notice that $\Omega \subseteq \Omega_n$, we have $\phi, \psi \in D_0^{1,2}(\Omega_n), \forall\, n$. Then

$$\begin{aligned} &\int_{\Omega_n} \nabla u_n \cdot \nabla \phi + \nabla v_n \cdot \nabla \psi \\ =&S_{\alpha,\beta,\lambda,\mu}(\Omega_n) \int_{\Omega_n} \Big(\lambda \frac{|u_n|^{2^*(s)-1}\phi}{|x|^s} + \mu \frac{|v_n|^{2^*(s)-1}\psi}{|x|^s} \\ &+ \kappa\alpha \frac{|u_n|^{\alpha-2}u_n\phi|v_n|^\beta}{|x|^s} + \kappa\beta \frac{|u_n|^\alpha |v_n|^{\beta-2}v_n\psi}{|x|^s}\Big) dx. \end{aligned} \tag{7.100}$$

Since $supp(\phi)$ $supp(\psi) \subseteq \Omega$, we have

$$\begin{aligned} &\int_{\Omega_n} \nabla u_n \cdot \nabla \phi + \nabla v_n \cdot \nabla \psi \\ =&S_{\alpha,\beta,\lambda,\mu}(\Omega_n) \int_{\Omega_n} \Big(\lambda \frac{|u_n|^{2^*(s)-1}\phi}{|x|^s} + \mu \frac{|v_n|^{2^*(s)-1}\psi}{|x|^s} \\ &+ \kappa\alpha \frac{|u_n|^{\alpha-2}u_n\phi|v_n|^\beta}{|x|^s} + \kappa\beta \frac{|u_n|^\alpha |v_n|^{\beta-2}v_n\psi}{|x|^s}\Big) dx. \text{ for all } n. \end{aligned} \tag{7.101}$$

Then apply the similar arguments as Corollary 7.2, we have

$$\begin{aligned} &\int_\Omega \nabla u \cdot \nabla \phi + \nabla v \cdot \nabla \psi \\ =&\bar{S} \int_\Omega \big(\lambda \frac{|u|^{2^*(s)-1}\phi}{|x|^s} + \mu \frac{|v|^{2^*(s)-1}\psi}{|x|^s} + \kappa\alpha \frac{|u|^{\alpha-2}u\phi|v|^\beta}{|x|^s} + \kappa\beta \frac{|u|^\alpha |v|^{\beta-2}v\psi}{|x|^s}\big) dx. \end{aligned} \tag{7.102}$$

Thereby the claim is proved. By (7.97) and $2 < 2^*(s)$, we have

$$\begin{aligned}
&S_{\alpha,\beta,\lambda,\mu}(\Omega)\\
\leq&\frac{\int_\Omega[|\nabla u|^2+|\nabla v|^2]dx}{\Big(\int_\Omega\big(\lambda\frac{|u|^{2^*(s)}}{|x|^s}+\mu\frac{|v|^{2^*(s)}}{|x|^s}+2^*(s)\kappa\frac{|u|^\alpha|v|^\beta}{|x|^s}\big)dx\Big)^{\frac{2}{2^*(s)}}}\\
\leq&\frac{\int_\Omega[|\nabla u|^2+|\nabla v|^2]dx}{\int_\Omega\big(\lambda\frac{|u|^{2^*(s)}}{|x|^s}+\mu\frac{|v|^{2^*(s)}}{|x|^s}+2^*(s)\kappa\frac{|u|^\alpha|v|^\beta}{|x|^s}\big)dx}=\bar{S}.
\end{aligned}\tag{7.103}$$

It follows from (7.96) and (7.103) that

$$S_{\alpha,\beta,\lambda,\mu}(\Omega)=\bar{S}=\lim_{n\to\infty}S_{\alpha,\beta,\lambda,\mu}(\Omega_n)$$

and

$$\int_\Omega\big(\lambda\frac{|u|^{2^*(s)}}{|x|^s}+\mu\frac{|v|^{2^*(s)}}{|x|^s}+2^*(s)\kappa\frac{|u|^\alpha|v|^\beta}{|x|^s}\big)dx=1.$$

Hence, (u,v) is an extremal function of $S_{\alpha,\beta,\lambda,\mu}(\Omega)$. The proof is completed. □

Define

$$\underline{S}_{\alpha,\beta,\lambda,\mu}:=\inf\big\{S_{\alpha,\beta,\lambda,\mu}(\Omega):\Omega\text{ is a cone properly contained in }\mathbb{R}^N\backslash\{0\}\big\}.\tag{7.104}$$

For any given unit versor ν in $\mathbb{R}^N$, let

$$\Omega_\theta:=\{x\in\mathbb{R}^N: x\cdot\nu>|x|\cos\theta\},\quad \theta\in(0,\pi].\tag{7.105}$$

Definition 7.1. *Assume $1<p<N, -\infty<t<N-p$, we denote by $D_t^{1,p}(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm*

$$\|u\|:=\Big(\int_\Omega\frac{|\nabla u|^p}{|x|^t}dx\Big)^{\frac{1}{p}}\tag{7.106}$$

Then we have the following result:

Proposition 7.4. *If F is a closed subset of a $k-$dimensional subspace of $\mathbb{R}^N$ with $k<N-t-p$, then $D_t^{1,p}(\Omega)=D_t^{1,p}(\Omega\backslash F)$. In particular, $D_t^{1,p}(\mathbb{R}^N)=D_t^{1,p}(\mathbb{R}^N\backslash\{0\})$ provided $N-t-p>0$.*

Proof. Without loss of generality, we assume that $\Omega=\mathbb{R}^N$. Notice that $C_c^\infty(\mathbb{R}^N\backslash F)\subseteq C_c^\infty(\mathbb{R}^N)$, by the definition, it is easy to see that $D_t^{1,p}(\mathbb{R}^N\backslash F)\subseteq D_t^{1,p}(\mathbb{R}^N)$.

On the other hand, for any $u\in D_t^{1,p}(\Omega)$, there exists a sequence $\{\varphi_n\}\subset C_c^\infty(\mathbb{R}^N)$ such that

$$\|\varphi_n-u\|^p=\int_{\mathbb{R}^N}\frac{|\nabla(\varphi_n-u)|^p}{|x|^t}dx\to 0\text{ as }n\to\infty.\tag{7.107}$$

If there exists a subsequence of $\{\varphi_{n_k}\}$ such that $supp(\varphi_{n_k})\cap F=\emptyset$, then $\{\varphi_{n_k}\}\subseteq C_c^\infty(\mathbb{R}^N\backslash F)$, and it follows that $u\in D_t^{1,p}(\mathbb{R}^N\backslash F)$ and the proof is completed. Hence,

we may assume that $supp(\varphi_n) \cap F \neq \emptyset$ for any n without loss of generality. Now, for any fixed n, we may choose a suitable cutoff function χ_δ such that $\chi_\delta = 0$ in $F_\delta, \chi_\delta = 0$ in $\mathbb{R}^N \backslash F_{2\delta}, \chi_\delta \in (0,1)$ in $(\mathbb{R}^N \backslash F_\delta) \cap F_{2\delta}, |\nabla \chi_\delta| \leq \frac{2}{\delta}$, where $F \subset \mathbb{R}^N$ and

$$F_\delta := \{x \in \mathbb{R}^N : dist(x, F) < \delta\}.$$

We note that $\chi_\delta \varphi_n \in C_c^\infty(\mathbb{R}^N \backslash F)$ for all $\delta > 0$.

Now, we estimate $\|\chi_\delta \varphi_n - \varphi_n\|^p$.

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{|\nabla(\chi_\delta \varphi_n - \varphi_n)|^p}{|x|^t} dx \\
=&\int_{\mathbb{R}^N} \frac{\left[|\nabla(\chi_\delta - 1)^2 \varphi_n^2| + 2\nabla(\chi_\delta - 1) \cdot \nabla \varphi_n (\chi_\delta - 1)\varphi_n + (\chi_\delta - 1)^2 |\nabla \varphi_n|^2\right]^{\frac{p}{2}}}{|x|^t} dx \\
\leq&\int_{\mathbb{R}^N} \frac{\left[2\left(|\nabla(\chi_\delta - 1)^2 \varphi_n^2| + (\chi_\delta - 1)^2 |\nabla \varphi_n|^2\right)\right]^{\frac{p}{2}}}{|x|^t} dx \\
\leq&2^p \int_{supp(\varphi_n) \cap F_{2\delta}} \frac{|\nabla \chi_\delta|^p}{|x|^t} |\varphi_n^p| dx + 2^p \int_{supp(\varphi_n)} |\chi_\delta - 1|^p \frac{|\nabla \varphi_n|^p}{|x|^t} dx \\
:=&I + II.
\end{aligned}$$

By the Lebesgue's dominated convergence theorem, it is easy to see that

$$II = 2^p \int_{supp(\varphi_n)} |\chi_\delta - 1|^p \frac{|\nabla \varphi_n|^p}{|x|^t} dx \to 0 \text{ as } \delta \to 0. \tag{7.108}$$

Recalling that $|\nabla \chi_\delta| \leq \frac{2}{\delta}$, there exists some $c_n > 0$ independent of δ such that

$$I = 2^p \int_{supp(\varphi_n) \cap F_{2\delta}} \frac{|\nabla \chi_\delta|^p}{|x|^\mu} |\varphi_n^p| dx \leq c_n (\frac{2}{\delta})^p \delta^{N-k-\mu}. \tag{7.109}$$

Hence, when $k < N - t - p$, we also have

$$I \to 0 \text{ as } \delta \to 0. \tag{7.110}$$

Hence, we can take some δ_n small enough such that

$$\|\chi_{\delta_n} \varphi_n - \varphi_n\|^p \leq \frac{1}{n}. \tag{7.111}$$

Now, we let $u_n := \chi_{\delta_n} \varphi_n \in C_c^\infty(\mathbb{R}^N \backslash F)$, we see that $\|u_n - u\|^p \to 0$ as $n \to \infty$. Hence, $u \in D_t^{1,p}(\mathbb{R}^N \backslash F)$. Thus, $D_t^{1,p}(\mathbb{R}^N) \subseteq D_t^{1,p}(\mathbb{R}^N \backslash F)$. Especially, when $N - t - p > 0$, take $k = 0$, we see that $D_t^{1,p}(\mathbb{R}^N) = D_t^{1,p}(\mathbb{R}^N \backslash \{0\})$ and the proof is completed.

□

Lemma 7.12.

$$S_{\alpha,\beta,\lambda,\mu}(\Omega_\pi) = S_{\alpha,\beta,\lambda,\mu}(\mathbb{R}^N) \textit{ for } N \geq 4.$$

Proof. Take $F = \mathbb{R}^N \backslash \Omega_\pi$, $F_n := F \cap \overline{B_n(0)}$. We note that F_n is a closed subset of a $1-$dimensional subspace of $\mathbb{R}^N$ and $\lim\limits_{n\to\infty} F_n = F$. Then by Proposition 7.4, $D_0^{1,2}(\mathbb{R}^N\backslash F_n) = D_0^{1,2}(\mathbb{R}^N)$ for any n. Then it follows that $D_0^{1,2}(\mathbb{R}^N\backslash F) = D_0^{1,2}(\mathbb{R}^N)$. That is, $D_0^{1,2}(\Omega_\pi) = D_0^{1,2}(\mathbb{R}^N)$. Hence, $S_{p,a,b}(\Omega_\pi) = S_{p,a,b}(\mathbb{R}^N)$. $\square$

Theorem 7.5. *For every $\tau \geq \underline{S}_{\alpha,\beta,\lambda,\mu}$, there exists a cone Ω in $\mathbb{R}^N$ such that $S_{\alpha,\beta,\lambda,\mu}(\Omega) = \tau$. Moreover, when $N \geq 4$, we have $\underline{S}_{\alpha,\beta,\lambda,\mu} = S_{\alpha,\beta,\lambda,\mu}(\mathbb{R}^N)$.*

Proof. Define a mapping $\tau : (0,\pi] \mapsto \mathbb{R}_+ \cup \{0\}$ with $\tau(\theta) = S_{\alpha,\beta,\lambda,\mu}(\Omega_\theta)$. Then by Remark 7.7, we see that the mapping τ is decreasing with related to θ. Evidently, $\tau(\theta)$ is continuous for a.e. $\theta \in (0,\pi]$. Furthermore, we can strengthen the conclusion. Indeed, let $\theta \in (0,\pi)$ be fixed. For any $\theta_n \uparrow \theta$, by (i) of Lemma 7.11, we have

$$\lim_{n\to\infty} \tau(\theta_n) = \tau(\theta). \tag{7.112}$$

On the other hand, for any $\theta_n \downarrow \theta$, by (ii) of Lemma 7.11, we also obtain (7.112). Hence, τ is continuous in $(0,\pi)$. In addition, $\tau(\theta_n) \downarrow S_{\alpha,\beta,\lambda,\mu}(\Omega_\pi) = \underline{S}_{\alpha,\beta,\lambda,\mu}$ as $\theta_n \uparrow \pi$ and $\tau(\theta_n) \uparrow +\infty$ as $\theta_n \downarrow 0$.

Especially, when $N \geq 4$, by Lemma 7.12, we have that

$$\underline{S}_{\alpha,\beta,\lambda,\mu} = S_{\alpha,\beta,\lambda,\mu}(\Omega_\pi) = S_{\alpha,\beta,\lambda,\mu}(\mathbb{R}^N).$$

$\square$

7.6 Existsence of infinitely many sign-changing solutions

In this subsection, we will study the existence of infinitely many sign-changing solutions as an application of Theorem 7.2.

Theorem 7.6. *Assume $s \in (0,2), \kappa > 0, \alpha > 1, \beta > 1, \alpha + \beta = 2^*(s)$. Let Ω_θ be defined by* (7.105) *for some fixed $\theta \in (0,\pi]$. Suppose that one of the following conditions holds:*

(a_1) $\lambda > \mu$ *and either* $1 < \beta < 2$ *or* $\begin{cases} \beta = 2 \\ \kappa > \frac{\lambda}{2^*(s)} \end{cases}$ *;*

(a_2) $\lambda = \mu$ *and either* $\min\{\alpha,\beta\} < 2$ *or* $\begin{cases} \min\{\alpha,\beta\} = 2, \\ \kappa > \frac{\lambda}{2^*(s)} \end{cases}$ *;*

(a_3) $\lambda < \mu$ *and either* $1 < \alpha < 2$ *or* $\begin{cases} \alpha = 2 \\ \kappa > \frac{\mu}{2^*(s)} \end{cases}$ *.*

Then the problem

$$\begin{cases} -\Delta u - \lambda \frac{1}{|x|^s}|u|^{2^*(s)-2}u = \kappa\alpha\frac{1}{|x|^s}|u|^{\alpha-2}u|v|^\beta & \textit{in } \Omega_\theta, \\ -\Delta v - \mu \frac{1}{|x|^s}|v|^{2^*(s)-2}v = \kappa\beta\frac{1}{|x|^s}|u|^{\alpha}|v|^{\beta-2}v & \textit{in } \Omega_\theta, \\ (u,v) \in \mathscr{D} := D_0^{1,2}(\Omega_\theta) \times D_0^{1,2}(\Omega_\theta), \end{cases} \tag{7.113}$$

possesses a sequence of sign changing solutions $\{(u_k, v_k)\}$ which are distinct under the modulo rotations around ν. Moreover, their energies c_k satisfies $\frac{c_k}{2^{k(N-1)}} \to +\infty$ as $k \to \infty$, where

$$
\begin{aligned}
c_k :=& \frac{1}{2}\int_{\Omega_\theta} [|\nabla u_k|^2 + |\nabla v_k|^2] dx \\
& - \frac{1}{2^*(s)}\int_{\Omega_\theta} \Big(\lambda \frac{|u_k|^{2^*(s)}}{|x|^s} + \mu \frac{|v_k|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa \frac{|u_k|^\alpha |v_k|^\beta}{|x|^s}\Big) dx. \qquad (7.114)
\end{aligned}
$$

Proof. The idea is inspired by [6]. We will construct a solution on Ω_θ by gluing together suitable signed solutions corresponding to each sub-cone. Using the spherical coordinates, we write $S^{n-1} = \{\theta_1, \cdots, \theta_{N-1} : \theta_i \in S^1, i = 1, \cdots, N-1\}$. For any fixed $k \in \mathbb{N}$, we set

$$\Sigma_j^{(k)} = (\frac{j}{2^{k-1}}\theta - \theta, \frac{j+1}{2^{k-1}}\theta - \theta) \quad j = 0, 1, \cdots, 2^k - 1$$

and for every choice of $(j_1, j_2, \cdots, j_{N-1}) \in \{0, 1, 2, \cdots, 2^k - 1\}^{N-1}$,

$$\Omega^{(k)}_{j_1,\cdots,j_{N-1}} := \{x \in \Omega_\theta : \frac{x}{|x|} \in \Sigma^{(k)}_{j_1} \times \cdots \times \Sigma^{(k)}_{j_1,\cdots,j_{N-1}}\}.$$

Due to Theorem 7.2, we can take $\big(u^{(k)}_{j_1,\cdots,j_{N-1}}, v^{(k)}_{j_1,\cdots,j_{N-1}}\big) \in D_0^{1,2}(\Omega^{(k)}_{j_1,\cdots,j_{N-1}}) \times D_0^{1,2}(\Omega^{(k)}_{j_1,\cdots,j_{N-1}})$ as the positive ground state solution to

$$
\begin{cases}
-\Delta u = S_{\alpha,\beta,\lambda,\beta}(\Omega^{(k)}_{j_1,\cdots,j_{N-1}})\Big(\lambda \frac{1}{|x|^s}|u|^{2^*(s)-2}u + \kappa\alpha \frac{1}{|x|^s}|u|^{\alpha-2}u|v|^\beta\Big) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega^{(k)}_{j_1,\cdots,j_{N-1}}, \\
-\Delta v = S_{\alpha,\beta,\lambda,\beta}(\Omega^{(k)}_{j_1,\cdots,j_{N-1}})\Big(\mu \frac{1}{|x|^s}|v|^{2^*(s)-2}v + \kappa\beta \frac{1}{|x|^s}|u|^{\alpha}|v|^{\beta-2}v\Big) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega^{(k)}_{j_1,\cdots,j_{N-1}}, \\
u \geq 0, v \geq 0, (u, v) \in \mathscr{D} := D_0^{1,2}(\Omega^{(k)}_{j_1,\cdots,j_{N-1}}) \times D_0^{1,2}(\Omega^{(k)}_{j_1,\cdots,j_{N-1}}).
\end{cases}
$$

We can extend every $u^{(k)}_{j_1,\cdots,j_{N-1}}$ and $v^{(k)}_{j_1,\cdots,j_{N-1}}$ outside $\Omega^{(k)}_{j_1,\cdots,j_{N-1}}$ by 0 and now we set

$$u^{(k)} := \sum_{j_1=0}^{2^k-1} \cdots \sum_{j_{N-1}=0}^{2^k-1} (-1)^{j_1+\cdots+j_{N-1}} u^{(k)}_{j_1,\cdots,j_{N-1}} \in D_0^{1,2}(\Omega_\theta)$$

and

$$v^{(k)} := \sum_{j_1=0}^{2^k-1} \cdots \sum_{j_{N-1}=0}^{2^k-1} (-1)^{j_1+\cdots+j_{N-1}} v^{(k)}_{j_1,\cdots,j_{N-1}} \in D_0^{1,2}(\Omega_\theta).$$

Notice that for any two different choices $(j_1, j_2, \cdots, j_{N-1}) \neq (\tilde{j}_1, \tilde{j}_2, \cdots, \tilde{j}_{N-1})$, there exists some rotation $R \in O(\mathbb{R}^N)$, the orthogonal transformation, such that

$$\Omega^{(k)}_{\tilde{j}_1,\cdots,\tilde{j}_{N-1}} = R\big(\Omega^{(k)}_{j_1,\cdots,j_{N-1}}\big).$$

Hence, we have

$$S_{\alpha,\beta,\lambda,\beta}(\Omega^{(k)}_{j_1,\cdots,j_{N-1}}) = S_{\alpha,\beta,\lambda,\beta}(\Omega^{(k)}_{\tilde{j}_1,\cdots,\tilde{j}_{N-1}}).$$

Then it follows that $(u^{(k)}, v^{(k)})$ weakly solves

$$\begin{cases} -\Delta u = S_{\alpha,\beta,\lambda,\beta}(\Omega^{(k)}_{j_1,\cdots,j_{N-1}})\Big(\lambda\frac{1}{|x|^s}|u|^{2^*(s)-2}u + \kappa\alpha\frac{1}{|x|^s}|u|^{\alpha-2}u|v|^{\beta}\Big) & \text{in } \Omega_\theta, \\ -\Delta v = S_{\alpha,\beta,\lambda,\beta}(\Omega^{(k)}_{j_1,\cdots,j_{N-1}})\Big(\mu\frac{1}{|x|^s}|v|^{2^*(s)-2}v + \kappa\beta\frac{1}{|x|^s}|u|^{\alpha}|v|^{\beta-2}v\Big) & \text{in } \Omega_\theta, \\ (u,v) \in \mathscr{D} := D_0^{1,2}(\Omega_\theta) \times D_0^{1,2}(\Omega_\theta). \end{cases}$$

Noting that $2^*(s) > 2$, after a scaling, let

$$u_k := \Big(S_{\alpha,\beta,\lambda,\beta}(\Omega^{(k)}_{j_1,\cdots,j_{N-1}})\Big)^{\frac{1}{2^*(s)-2}} u^{(k)},\ v_k := \Big(S_{\alpha,\beta,\lambda,\beta}(\Omega^{(k)}_{j_1,\cdots,j_{N-1}})\Big)^{\frac{1}{2^*(s)-2}} v^{(k)}$$

then (u_k, v_k) weakly solves (7.113). By the construction of u_k and v_k, it is easy to see that (u_k, v_k) is a sign changing solution and $\{(u_k, v_k)\}$ are distinct under modulo rotations around ν.

Moveover, we have

$$\begin{aligned} c_k :=& \frac{1}{2}\int_{\Omega_\theta}[|\nabla u_k|^2 + |\nabla v_k|^2]dx \\ &- \frac{1}{2^*(s)}\int_{\Omega_\theta}\Big(\lambda\frac{|u_k|^{2^*(s)}}{|x|^s} + \mu\frac{|v_k|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa\frac{|u_k|^{\alpha}|v_k|^{\beta}}{|x|^s}\Big)dx \\ =&(\frac{1}{2} - \frac{1}{2^*(s)})\int_{\Omega_\theta}[|\nabla u_k|^2 + |\nabla v_k|^2]dx \\ =&(\frac{1}{2} - \frac{1}{2^*(s)})\Big(S_{\alpha,\beta,\lambda,\mu}(\Omega^{(k)}_{j_1,\cdots,j_{N-1}})\Big)^{\frac{2}{2^*(s)-2}}\int_{\Omega_\theta}[|\nabla u^{(k)}|^2 + |\nabla v^{(k)}|^2]dx \\ =&(\frac{1}{2} - \frac{1}{2^*(s)})\Big(S_{\alpha,\beta,\lambda,\mu}(\Omega^{(k)}_{j_1,\cdots,j_{N-1}})\Big)^{\frac{2}{2^*(s)-2}}\sum_{j_1=0}^{2^k-1}\cdots\sum_{j_{N-1}=0}^{2^k-1}\int_{\Omega^{(k)}_{j_1,\cdots,j_{N-1}}} \\ &[|\nabla u^{(k)}_{j_1,\cdots,j_{N-1}}|^2 + |\nabla v^{(k)}_{j_1,\cdots,j_{N-1}}|^2]dx \\ =&(\frac{1}{2} - \frac{1}{2^*(s)})\Big(S_{\alpha,\beta,\lambda,\mu}(\Omega^{(k)}_{j_1,\cdots,j_{N-1}})\Big)^{\frac{2}{2^*(s)-2}} 2^{k(N-1)} \\ &\cdot\int_{\Omega_{0,\cdots,0}}[|\nabla u^{(k)}_{0,\cdots,0}|^2 + |\nabla v^{(k)}_{0,\cdots,0}|^2]dx \\ =&(\frac{1}{2} - \frac{1}{2^*(s)})\Big(S_{\alpha,\beta,\lambda,\mu}(\Omega^{(k)}_{j_1,\cdots,j_{N-1}})\Big)^{\frac{2}{2^*(s)-2}} 2^{k(N-1)}. \end{aligned}$$

By Lemma 7.11, $S_{\alpha,\beta,\lambda,\mu}(\Omega^{(k)}_{j_1,\cdots,j_{N-1}}) \to +\infty$ as $k \to \infty$. Recalling that $2^*(s) > 2$ again, we obtain that

$$\frac{c_k}{2^{k(N-1)}} \to +\infty.$$

□

Apply the same argument as in the proof of Theorem 7.6, we can obtain the following result for the system defined on $\mathbb{R}^N$:

Theorem 7.7. *Assume* $s \in (0,2), N \geq 4, \kappa > 0, \alpha > 1, \beta > 1, \alpha + \beta = 2^*(s)$. *Suppose that one of the following conditions holds:*

(a_1) $\lambda > \mu$ *and either* $1 < \beta < 2$ *or* $\begin{cases} \beta = 2 \\ \kappa > \frac{\lambda}{2^*(s)} \end{cases}$ *;*

(a_2) $\lambda = \mu$ *and either* $\min\{\alpha, \beta\} < 2$ *or* $\begin{cases} \min\{\alpha, \beta\} = 2, \\ \kappa > \frac{\lambda}{2^*(s)} \end{cases}$ *;*

(a_3) $\lambda < \mu$ *and either* $1 < \alpha < 2$ *or* $\begin{cases} \alpha = 2 \\ \kappa > \frac{\mu}{2^*(s)} \end{cases}$ *.*

Then the problem

$$\begin{cases} -\Delta u - \lambda \frac{1}{|x|^s} |u|^{2^*(s)-2} u = \kappa \alpha \frac{1}{|x|^s} |u|^{\alpha-2} u |v|^\beta & \text{in } \mathbb{R}^N, \\ -\Delta v - \mu \frac{1}{|x|^s} |v|^{2^*(s)-2} v = \kappa \beta \frac{1}{|x|^s} |u|^\alpha |v|^{\beta-2} v & \text{in } \mathbb{R}^N, \\ (u,v) \in \mathscr{D} := D_0^{1,2}(\mathbb{R}^N) \times D_0^{1,2}(\mathbb{R}^N), \end{cases} \tag{7.115}$$

possesses a sequence of sign changing solutions $\{(u_k, v_k)\}$ *whose energies* $\dfrac{c_k}{2^{k(N-1)}} \to +\infty$ *as* $k \to \infty$*, where*

$$\begin{aligned} c_k := &\frac{1}{2} \int_{\mathbb{R}^N} [|\nabla u_k|^2 + |\nabla v_k|^2] dx \\ &- \frac{1}{2^*(s)} \int_{\mathbb{R}^N} \Big(\lambda \frac{|u_k|^{2^*(s)}}{|x|^s} + \mu \frac{|v_k|^{2^*(s)}}{|x|^s} + 2^*(s) \kappa \frac{|u_k|^\alpha |v_k|^\beta}{|x|^s} \Big) dx. \end{aligned}$$

Proof. It is a straightforward consequence of Theorems 7.6 and 7.5. We just keep in mind that when $N \geq 4$, we have that $S_{\alpha,\beta,\lambda,\mu}(\Omega_\pi) = S_{\alpha,\beta,\lambda,\mu}(\mathbb{R}^N)$. □

Remark 7.8. *It is clear that this kind arguments used in the proof of Theorem 7.6 can be adapted to other cones with suitable symmetry.*

7.7 Further results on more general domain Ω and on the sharp constant $S_{\alpha,\beta,\lambda,\mu}(\Omega)$

Remark 7.9. *Given a general open domain* Ω *(not necessarily a cone), we let* $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ *be defined by* (7.2) *if* $\Omega \neq \emptyset$ *and* $S_{\alpha,\beta,\lambda,\mu}(\emptyset) = +\infty$*. In this subsection, we are concerned with whether* $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ *can be achieved or not and we give some operational way to compute the value of* $S_{\alpha,\beta,\lambda,\mu}(\Omega)$*.*

We note that Ω can be written as a union of a sequence of domains, $\Omega = \bigcup_{n=1}^\infty \Omega_n$.

Lemma 7.13. *Assume $\Omega_i \cap \Omega_j = \emptyset \ \forall \ i \neq j$, then we have*

$$S_{\alpha,\beta,\lambda,\mu}(\Omega) = \inf_{n\geq 1} S_{\alpha,\beta,\lambda,\mu}(\Omega_n).$$

Proof. For any n, since $\Omega_n \subseteq \Omega$, we have

$$S_{\alpha,\beta,\lambda,\mu}(\Omega_n) \geq S_{\alpha,\beta,\lambda,\mu}(\Omega) \text{ for all } n.$$

Hence,

$$\inf_{n\geq 1} S_{\alpha,\beta,\lambda,\mu}(\Omega_n) \geq S_{\alpha,\beta,\lambda,\mu}(\Omega). \tag{7.116}$$

On the other hand, for any $\varepsilon > 0$, there exists a pair (u, v) such that

$$\int_\Omega \big(\lambda \frac{|u|^{2^*(s)}}{|x|^s} + \mu \frac{|v|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa \frac{|u|^\alpha |v|^\beta}{|x|^s}\big) dx = 1 \tag{7.117}$$

and

$$\int_\Omega [|\nabla u|^2 + |\nabla v|^2] dx < S_{\alpha,\beta,\lambda,\mu}(\Omega) + \varepsilon. \tag{7.118}$$

Set $u_n = u\big|_{\Omega_n}, v_n = v\big|_{\Omega_n}$, since $\Omega_i \cap \Omega_j = \emptyset$ for all $i \neq j$, we have $(u_n, v_n) \in D_0^{1,2}(\Omega_n) \times D_0^{1,2}(\Omega_n)$ and $u = \sum_{n=1}^\infty u_n, v = \sum_{n=1}^\infty v_n$. Then

$$\begin{aligned}
&\int_{\Omega_n} [|\nabla u_n|^2 + |\nabla v_n|^2] dx \\
\geq& S_{\alpha,\beta,\lambda,\mu}(\Omega_n) \Big(\int_{\Omega_n} \big(\lambda \frac{|u_n|^{2^*(s)}}{|x|^s} + \mu \frac{|v_n|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa \frac{|u_n|^\alpha |v_n|^\beta}{|x|^s}\big) dx \Big)^{\frac{2}{2^*(s)}} \\
\geq& \big(\inf_{n\geq 1} S_{\alpha,\beta,\lambda,\mu}(\Omega_n) \big) \Big(\int_{\Omega_n} \big(\lambda \frac{|u_n|^{2^*(s)}}{|x|^s} + \mu \frac{|v_n|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa \frac{|u_n|^\alpha |v_n|^\beta}{|x|^s}\big) dx \Big)^{\frac{2}{2^*(s)}} \\
\geq& \big(\inf_{n\geq 1} S_{\alpha,\beta,\lambda,\mu}(\Omega_n) \big) \int_{\Omega_n} \big(\lambda \frac{|u_n|^{2^*(s)}}{|x|^s} + \mu \frac{|v_n|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa \frac{|u_n|^\alpha |v_n|^\beta}{|x|^s}\big) dx,
\end{aligned}$$

here we use $\frac{2}{2^*(s)} < 1$ and $\int_{\Omega_n} \big(\lambda \frac{|u_n|^{2^*(s)}}{|x|^s} + \mu \frac{|v_n|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa \frac{|u_n|^\alpha |v_n|^\beta}{|x|^s}\big) dx \leq 1$. It follows that

$$\begin{aligned}
&\int_\Omega [|\nabla u|^2 + |\nabla v|^2] dx \\
=& \sum_{n=1}^\infty \int_{\Omega_n} [|\nabla u_n|^2 + |\nabla v_n|^2] dx \\
\geq& \big(\inf_{n\geq 1} S_{\alpha,\beta,\lambda,\mu}(\Omega_n) \big) \sum_{n=1}^\infty \int_{\Omega_n} \big(\lambda \frac{|u_n|^{2^*(s)}}{|x|^s} + \mu \frac{|v_n|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa \frac{|u_n|^\alpha |v_n|^\beta}{|x|^s}\big) dx \\
=& \big(\inf_{n\geq 1} S_{\alpha,\beta,\lambda,\mu}(\Omega_n) \big) \int_\Omega \big(\lambda \frac{|u|^{2^*(s)}}{|x|^s} + \mu \frac{|v|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa \frac{|u|^\alpha |v|^\beta}{|x|^s}\big) dx \\
=& \inf_{n\geq 1} S_{\alpha,\beta,\lambda,\mu}(\Omega_n).
\end{aligned}$$

Hence, $\inf_{n\geq 1} S_{\alpha,\beta,\lambda,\mu}(\Omega_n) \leq S_{\alpha,\beta,\lambda,\mu}(\Omega) + \varepsilon$. Therefore,

$$\inf_{n\geq 1} S_{\alpha,\beta,\lambda,\mu}(\Omega_n) \leq S_{\alpha,\beta,\lambda,\mu}(\Omega). \tag{7.119}$$

By (7.116) and (7.119), we complete the proof. □

Next, for $r > 0$, we set

$$\Omega_r := \Omega \cap B_r, \quad \Omega^r := \Omega \backslash B_r. \tag{7.120}$$

By Remark 7.7, we see that the mapping $r \mapsto S_{\alpha,\beta,\lambda,\mu}(\Omega_r)$ is non increasing and the mapping $r \mapsto S_{\alpha,\beta,\lambda,\mu}(\Omega^r)$ is non decreasing. Hence, we can define

$$S^0_{\alpha,\beta,\lambda,\mu}(\Omega) := \lim_{r\to 0} S_{\alpha,\beta,\lambda,\mu}(\Omega_r)$$

and

$$S^\infty_{\alpha,\beta,\lambda,\mu}(\Omega) := \lim_{r\to \infty} S_{\alpha,\beta,\lambda,\mu}(\Omega^r).$$

Remark 7.10. *It is easy to see that $S^0_{\alpha,\beta,\lambda,\mu}(\Omega)$ and $S^\infty_{\alpha,\beta,\lambda,\mu}(\Omega)$ still have the monotonicity property. Precisely, if $\Omega_1 \subseteq \Omega_2$, then we have*

$$S^0_{\alpha,\beta,\lambda,\mu}(\Omega_1) \geq S^0_{\alpha,\beta,\lambda,\mu}(\Omega_2), \quad S^\infty_{\alpha,\beta,\lambda,\mu}(\Omega_1) \geq S^\infty_{\alpha,\beta,\lambda,\mu}(\Omega_2).$$

Theorem 7.8. *Assume that $s \in (0,2), \kappa > 0, \alpha > 1, \beta > 1, \alpha+\beta = 2^*(s)$ and Ω is an open domain of $\mathbb{R}^N$. Let $\{(u_n, v_n)\}$ be a minimizing sequence, i.e.,*

$$\int_\Omega \big(\lambda\frac{|u_n|^{2^*(s)}}{|x|^s} + \mu\frac{|v_n|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa\frac{|u_n|^\alpha|v_n|^\beta}{|x|^s}\big)dx \equiv 1$$

and

$$\int_\Omega [|\nabla u_n|^2 + |\nabla v_n|^2]dx \to S_{\alpha,\beta,\lambda,\mu}(\Omega)$$

as $n \to \infty$. Then one of the following cases happens:

(a) There exists some $(u,v) \in D^{1,2}_0(\Omega) \times D^{1,2}_0(\Omega)$ such that $u_n \to u, v_n \to v$ strongly in $D^{1,2}_0(\Omega)$ and $(u,v) \neq (0,0)$ is an extremal function of $S_{\alpha,\beta,\lambda,\mu}(\Omega)$;

(b) Going to a subsequence if necessary, we set

$$\eta := \lim_{r\to 0}\lim_{n\to\infty}\int_{\Omega_r} \big(\lambda\frac{|u_n|^{2^*(s)}}{|x|^s} + \mu\frac{|v_n|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa\frac{|u_n|^\alpha|v_n|^\beta}{|x|^s}\big)dx.$$

Then

$$S^0_{\alpha,\beta,\lambda,\mu}(\Omega)\eta^{\frac{2}{2^*(s)}} + S^\infty_{\alpha,\beta,\lambda,\mu}(\Omega)(1-\eta)^{\frac{2}{2^*(s)}} \leq S_{\alpha,\beta,\lambda,\mu}(\Omega). \tag{7.121}$$

Proof. It is easy to see that (u,v) is an extremal function of $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ if and only if (u,v) is a ground state solution of

$$\begin{cases}-\Delta u = S_{\alpha,\beta,\lambda,\beta}(\Omega)\Big(\lambda\frac{1}{|x|^s}|u|^{2^*(s)-2}u+\kappa\alpha\frac{1}{|x|^s}|u|^{\alpha-2}u|v|^{\beta}\Big) & \text{in } \Omega,\\ -\Delta v = S_{\alpha,\beta,\lambda,\beta}(\Omega)\Big(\mu\frac{1}{|x|^s}|v|^{2^*(s)-2}v+\kappa\beta\frac{1}{|x|^s}|u|^{\alpha}|v|^{\beta-2}v\Big) & \text{in } \Omega,\\ u\geq 0, v\geq 0, (u,v)\in \mathscr{D}:=D_0^{1,2}(\Omega)\times D_0^{1,2}(\Omega), \end{cases} \tag{7.122}$$

Since $\{(u_n,v_n)\}$ is a minimizing sequence, we have that $\{(u_n,v_n)\}$ is a bounded $(PS)_d$ sequence with $d=(\frac{1}{2}-\frac{1}{2^*(s)})S_{\alpha,\beta,\lambda,\beta}(\Omega)$. Without loss of generality, we assume that $u_n\rightharpoonup u, v_n\rightharpoonup v$ in $D_0^{1,2}(\Omega)$ and $u_n\to u, v_n\to v$ a.e. in Ω. Then it is easy to see that (u,v) is a weak solution to (7.122) and

$$0\leq \int_\Omega \big(\lambda\frac{|u|^{2^*(s)}}{|x|^s}+\mu\frac{|v|^{2^*(s)}}{|x|^s}+2^*(s)\kappa\frac{|u|^{\alpha}|v|^{\beta}}{|x|^s}\big)dx\leq 1,$$

$$\int_\Omega [|\nabla u|^2+|\nabla v|^2]dx\leq S_{\alpha,\beta,\lambda,\mu}(\Omega).$$

Case 1: If $(u,v)\neq(0,0)$, we shall prove that (a) happens. In this case, (u,v) is a nontrivial solution or semi-trivial solution of (7.122). We claim

$$\int_\Omega \big(\lambda\frac{|u|^{2^*(s)}}{|x|^s}+\mu\frac{|v|^{2^*(s)}}{|x|^s}+2^*(s)\kappa\frac{|u|^{\alpha}|v|^{\beta}}{|x|^s}\big)dx=1.$$

If not, $0<\int_\Omega \big(\lambda\frac{|u|^{2^*(s)}}{|x|^s}+\mu\frac{|v|^{2^*(s)}}{|x|^s}+2^*(s)\kappa\frac{|u|^{\alpha}|v|^{\beta}}{|x|^s}\big)dx<1$. Then

$$\begin{aligned}&S_{\alpha,\beta,\lambda,\mu}(\Omega)\\ =&\frac{\int_\Omega[|\nabla u|^2+|\nabla v|^2]dx}{\int_\Omega \big(\lambda\frac{|u|^{2^*(s)}}{|x|^s}+\mu\frac{|v|^{2^*(s)}}{|x|^s}+2^*(s)\kappa\frac{|u|^{\alpha}|v|^{\beta}}{|x|^s}\big)dx}\\ >&\frac{\int_\Omega[|\nabla u|^2+|\nabla v|^2]dx}{\Big(\int_\Omega \big(\lambda\frac{|u|^{2^*(s)}}{|x|^s}+\mu\frac{|v|^{2^*(s)}}{|x|^s}+2^*(s)\kappa\frac{|u|^{\alpha}|v|^{\beta}}{|x|^s}\big)dx\Big)^{\frac{2}{2^*(s)}}}\\ \geq& S_{\alpha,\beta,\lambda,\mu}(\Omega),\end{aligned}$$

a contradiction. Hence,

$$\int_\Omega \big(\lambda\frac{|u|^{2^*(s)}}{|x|^s}+\mu\frac{|v|^{2^*(s)}}{|x|^s}+2^*(s)\kappa\frac{|u|^{\alpha}|v|^{\beta}}{|x|^s}\big)dx=1,$$

$$\int_\Omega [|\nabla u|^2+|\nabla v|^2]dx= S_{\alpha,\beta,\lambda,\mu}(\Omega).$$

That is, (u,v) is an extremal function of $S_{\alpha,\beta,\lambda,\mu}(\Omega)$. Note that $\|u_n-u\|=\|u_n\|-\|u\|+o(1), \|v_n-v\|=\|v_n\|-\|v\|+o(1)$, we see that $u_n\to u, v_n\to v$ in $D_0^{1,2}(\Omega)$.

Case 2: If $(u,v)=(0,0)$, we shall prove that (b) happens. The idea is similar to the proof of Lemma 7.7 and Corollary 7.2. Going to a subsequence if necessary, we set

$$\Lambda_0 := \lim_{r\to 0}\lim_{n\to\infty}\int_{\Omega_r}[|\nabla u_n|^2+|\nabla v_n|^2]dx$$

and

$$\Lambda^\infty := \lim_{r\to \infty}\lim_{n\to\infty}\int_{\Omega^r}[|\nabla u_n|^2+|\nabla v_n|^2]dx.$$

Recalling the Rellich-Kondrachov compact theorem and $2<2^*(s)<2^*:=\frac{2N}{N-2}$, we have $(u_n,v_n)\to(0,0)$ in $L^t_{loc}(\Omega)\times L^t_{loc}(\Omega)$ for all $1<t<2^*$. Hence,

$$\int_{\tilde\Omega}\frac{|u_n|^{2^*(s)}}{|x|^s}dx=o(1),\int_{\tilde\Omega}\frac{|v_n|^{2^*(s)}}{|x|^s}dx=o(1) \tag{7.123}$$

for any bounded domain $\tilde\Omega\subset\Omega$ such that $0\notin\overline{\tilde\Omega}$. Hence, we obtain that

$$\lim_{r\to\infty}\lim_{n\to\infty}\int_{\Omega^r}\big(\lambda\frac{|u_n|^{2^*(s)}}{|x|^s}+\mu\frac{|v_n|^{2^*(s)}}{|x|^s}+2^*(s)\kappa\frac{|u_n|^\alpha|v_n|^\beta}{|x|^s}\big)dx=1-\eta. \tag{7.124}$$

Similar to the formula (7.52), we have

$$S^0_{\alpha,\beta,\lambda,\mu}(\Omega)\eta^{\frac{2}{2^*(s)}}\le\Lambda_0. \tag{7.125}$$

and similar to the formula (7.53), we have

$$S^\infty_{\alpha,\beta,\lambda,\mu}(\Omega)\big(1-\eta\big)^{\frac{2}{2^*(s)}}\le\Lambda^\infty. \tag{7.126}$$

Then by (7.125) and (7.126), we have

$$S^0_{\alpha,\beta,\lambda,\mu}(\Omega)\eta^{\frac{2}{2^*(s)}}+S^\infty_{\alpha,\beta,\lambda,\mu}(\Omega)\big(1-\eta\big)^{\frac{2}{2^*(s)}}\le\Lambda_0+\Lambda^\infty\le S_{\alpha,\beta,\lambda,\mu}(\Omega). \tag{7.127}$$

□

Theorem 7.8 is a kind of concentration compactness principle, original spirit we refer to [21]. Basing on Theorem 7.8, we have the following using result:

Corollary 7.4. *Assume that $s\in(0,2),\kappa>0,\alpha>1,\beta>1,\alpha+\beta=2^*(s)$ and Ω is an open domain of $\mathbb{R}^N$. Then we always have*

$$S_{\alpha,\beta,\lambda,\mu}(\Omega)\le min\big\{S^0_{\alpha,\beta,\lambda,\mu}(\Omega),\ \ S^\infty_{\alpha,\beta,\lambda,\mu}(\Omega)\big\}. \tag{7.128}$$

Moreover, if $S_{\alpha,\beta,\lambda,\mu}(\Omega)<min\{S^0_{\alpha,\beta,\lambda,\mu}(\Omega),S^\infty_{\alpha,\beta,\lambda,\mu}(\Omega)\}$, then $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ can be achieved.

Proof. We note that $\Omega_r\subseteq\Omega,\Omega^r\subseteq\Omega$, by the monotonicity property, for any $r>0$, we have

$$S_{\alpha,\beta,\lambda,\mu}(\Omega)\le S_{\alpha,\beta,\lambda,\mu}(\Omega_r),\quad S_{\alpha,\beta,\lambda,\mu}(\Omega)\le S_{\alpha,\beta,\lambda,\mu}(\Omega^r)$$

which deduce (7.128). Moreover, if $S_{\alpha,\beta,\lambda,\mu}(\Omega)<min\{S^0_{\alpha,\beta,\lambda,\mu}(\Omega),S^\infty_{\alpha,\beta,\lambda,\mu}(\Omega)\}$, then (7.121) will never meet. Hence, only case (a) of Theorem7.8 happens. Thus, $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ is achieved. □

Furthermore, we have the following result:

Corollary 7.5. *Assume that* $s \in (0,2), \kappa > 0, \alpha > 1, \beta > 1, \alpha+\beta = 2^*(s)$ *and* Ω *is an open domain of* $\mathbb{R}^N$*. If* $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ *can not be achieved, then at least one of the following holds:*

(i) $S_{\alpha,\beta,\lambda,\mu}(\Omega) = S_{\alpha,\beta,\lambda,\mu}(\Omega_r)$ *for* $\forall\ r > 0$.

(ii) $S_{\alpha,\beta,\lambda,\mu}(\Omega) = S_{\alpha,\beta,\lambda,\mu}(\Omega^r)$ *for* $\forall\ r > 0$.

Proof. When $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ is not attained, by Corollary 7.4, we have

$$S_{\alpha,\beta,\lambda,\mu}(\Omega) = min\{S^0_{\alpha,\beta,\lambda,\mu}(\Omega), S^\infty_{\alpha,\beta,\lambda,\mu}(\Omega)\}.$$

Without loss of generality, we assume that $S^0_{\alpha,\beta,\lambda,\mu}(\Omega) \le S^\infty_{\alpha,\beta,\lambda,\mu}(\Omega)$, then we have

$$S_{\alpha,\beta,\lambda,\mu}(\Omega) = S^0_{\alpha,\beta,\lambda,\mu}(\Omega).$$

Next, we shall prove that case (i) holds. If not, assume that there exists some $r_0 > 0$ such that $S_{\alpha,\beta,\lambda,\mu}(\Omega) \ne S_{\alpha,\beta,\lambda,\mu}(\Omega_{r_0})$, then by the monotonicity property, we have $S_{\alpha,\beta,\lambda,\mu}(\Omega) < S_{\alpha,\beta,\lambda,\mu}(\Omega_{r_0}) \le S^0_{\alpha,\beta,\lambda,\mu}(\Omega_{r_0}) = S^0_{\alpha,\beta,\lambda,\mu}(\Omega)$, a contradiction. ☐

Remark 7.11. *We note that the inverse statement of Corollary 7.5 is not true. For example, by Theorem 7.1, when* Ω *is a cone,* $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ *is attained provided* $1 < \alpha, 1 < \beta, \alpha+\beta = 2^*(s), s \in (0,2), \kappa > 0$*. However, we still have the following result.*

Lemma 7.14. *Assume that* $s \in (0,2), \kappa > 0, \alpha > 1, \beta > 1, \alpha+\beta = 2^*(s)$*, then*

$$S_{\alpha,\beta,\lambda,\mu}(\Omega) = S_{\alpha,\beta,\lambda,\mu}(\Omega_r) = S_{\alpha,\beta,\lambda,\mu}(\Omega^r) \text{ for any } r > 0 \text{ if } \Omega \text{ is a cone of } \mathbb{R}^N.$$

In particular,

$$S_{\alpha,\beta,\lambda,\mu}(\mathbb{R}^N) = S_{\alpha,\beta,\lambda,\mu}(B_r) = S_{\alpha,\beta,\lambda,\mu}(\mathbb{R}^N\backslash B_r) \text{ for any } r > 0.$$

Further, $S_{\alpha,\beta,\lambda,\mu}(\Omega) = S_{\alpha,\beta,\lambda,\mu}(\mathbb{R}^N)$ *provided that either* Ω *is a general open domain with* $0 \in \Omega$ *or* Ω *is an exterior domain.*

Furthermore, let A *be a cone of* $\mathbb{R}^N$*, and* $\Omega = A\backslash F$*, where* F *is a closed subset of* A *such that* $0 \notin F$ *or* F *is bounded. Then*

$$S_{\alpha,\beta,\lambda,\mu}(\Omega) = S_{\alpha,\beta,\lambda,\mu}(A).$$

Proof. We only prove $S_{\alpha,\beta,\lambda,\mu}(\Omega) = S_{\alpha,\beta,\lambda,\mu}(\Omega_r)$ and the others are similar. By the monotonicity property, we see that $S_{\alpha,\beta,\lambda,\mu}(\Omega) \le S_{\alpha,\beta,\lambda,\mu}(\Omega_r)$. Next, we shall prove the opposite inequality. By Theorem 7.1, $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ is attained. Now, let $(u,v) \in D^{1,2}_0(\Omega) \times D^{1,2}_0(\Omega)$ be an extremal function of $S_{\alpha,\beta,\lambda,\mu}(\Omega)$. We also let $\chi_\rho(x) \in C^\infty_c(\mathbb{R}^N)$ be a cut-off function such that $\chi_\rho(x) \equiv 1$ in $B_{\frac{\rho}{2}}$, $\chi_\rho(x) \equiv 0$ in

$\mathbb{R}^N\backslash B_\rho, |\nabla\chi_\rho(x)| \leq \frac{4}{\rho}$ and define $\phi_\rho := \chi_\rho(x)u(x), \psi_\rho := \chi_\rho(x)v(x) \in C_c^\infty(\Omega_\rho)$. It is easy to see that

$$\int_\Omega [|\nabla\phi_\rho|^2 + |\nabla\psi_\rho|^2]dx \to \int_\Omega [|\nabla u|^2 + |\nabla v|^2]dx = S_{\alpha,\beta,\lambda,\mu}(\Omega),$$

and

$$\begin{aligned}
&\int_\Omega \big(\lambda\frac{|\phi_\rho|^{2^*(s)}}{|x|^s} + \mu\frac{|\psi_\rho|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa\frac{|\phi_\rho|^\alpha|\psi_\rho|^\beta}{|x|^s}\big)dx\\
\to&\int_\Omega \big(\lambda\frac{|u|^{2^*(s)}}{|x|^s} + \mu\frac{|v|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa\frac{|u|^\alpha|v|^\beta}{|x|^s}\big)dx\\
=&1
\end{aligned}$$

as $\rho \to +\infty$. Then $\forall\, \varepsilon > 0$, there exists some $\rho_0 > 0$ such that

$$\frac{\int_\Omega [|\nabla\phi_{\rho_0}|^2 + |\nabla\psi_{\rho_0}|^2]dx}{\Big(\int_\Omega \big(\lambda\frac{|\phi_{\rho_0}|^{2^*(s)}}{|x|^s} + \mu\frac{|\psi_{\rho_0}|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa\frac{|\phi_{\rho_0}|^\alpha|\psi_{\rho_0}|^\beta}{|x|^s}\big)dx\Big)^{\frac{2}{2^*(s)}}} \leq S_{\alpha,\beta,\lambda,\mu}(\Omega) + \varepsilon.$$

Now, consider $\tilde{u}_r(x) := \phi_{\rho_0}(\frac{\rho_0}{r}x), \tilde{v}_r(x) := \psi_{\rho_0}(\frac{\rho_0}{r}x) \in C_c^\infty(\Omega_r)$ and

$$\begin{aligned}
S_{p,a,b}(\Omega_r) \leq&\frac{\int_{\Omega_r} [|\nabla\tilde{u}_r(x)|^2 + |\nabla\tilde{v}_r(x)|^2]dx}{\Big(\int_{\Omega_r} \big(\lambda\frac{|\tilde{u}_r(x)|^{2^*(s)}}{|x|^s} + \mu\frac{|\tilde{v}_r(x)|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa\frac{|\tilde{u}_r(x)|^\alpha|\tilde{v}_r(x)|^\beta}{|x|^s}\big)dx\Big)^{\frac{2}{2^*(s)}}}\\
=&\frac{\int_\Omega [|\nabla\phi_{\rho_0}|^2 + |\nabla\psi_{\rho_0}|^2]dx}{\Big(\int_\Omega \big(\lambda\frac{|\phi_{\rho_0}|^{2^*(s)}}{|x|^s} + \mu\frac{|\psi_{\rho_0}|^{2^*(s)}}{|x|^s} + 2^*(s)\kappa\frac{|\phi_{\rho_0}|^\alpha|\psi_{\rho_0}|^\beta}{|x|^s}\big)dx\Big)^{\frac{2}{2^*(s)}}}\\
\leq& S_{\alpha,\beta,\lambda,\mu}(\Omega) + \varepsilon.
\end{aligned}$$

Hence,

$$S_{\alpha,\beta,\lambda,\mu}(\Omega_r) \leq S_{\alpha,\beta,\lambda,\mu}(\Omega).$$

Especially, take $\Omega = \mathbb{R}^N$, we see that

$$S_{\alpha,\beta,\lambda,\mu}(\mathbb{R}^N) = S_{\alpha,\beta,\lambda,\mu}(B_r) = S_{\alpha,\beta,\lambda,\mu}(\mathbb{R}^N\backslash B_r) \text{ for any } r > 0.$$

Hence, when $0 \in \Omega$, then there exists some $r > 0$ such that $B_r \subset \Omega$, then

$$S_{\alpha,\beta,\lambda,\mu}(B_r) \geq S_{\alpha,\beta,\lambda,\mu}(\Omega) \geq S_{\alpha,\beta,\lambda,\mu}(\mathbb{R}^N) = S_{\alpha,\beta,\lambda,\mu}(B_r).$$

If Ω is an exterior domain, there exists some $r > 0$ such that $(\mathbb{R}^N\backslash B_r) \subset \Omega$, by the monotonicity property again, we have

$$S_{\alpha,\beta,\lambda,\mu}(\mathbb{R}^N\backslash B_r) \geq S_{\alpha,\beta,\lambda,\mu}(\Omega) \geq S_{\alpha,\beta,\lambda,\mu}(\mathbb{R}^N) = S_{\alpha,\beta,\lambda,\mu}(\mathbb{R}^N\backslash B_r).$$

Furthermore, A is a cone and $\Omega = A\backslash F \subset A$, then we have $S_{\alpha,\beta,\lambda,\mu}(\Omega) \geq S_{\alpha,\beta,\lambda,\mu}(A)$. If $0 \notin F$, then there exists some $r > 0$ such that $\Omega_r = A_r$, then

$$S_{\alpha,\beta,\lambda,\mu}(\Omega) \leq S_{\alpha,\beta,\lambda,\mu}(\Omega_r) = S_{\alpha,\beta,\lambda,\mu}(A_r) = S_{\alpha,\beta,\lambda,\mu}(A).$$

Hence, we have $S_{\alpha,\beta,\lambda,\mu}(\Omega) = S_{\alpha,\beta,\lambda,\mu}(A)$. If F is bounded, then there exists some $r > 0$ such that $\Omega^r = A^r$, then

$$S_{\alpha,\beta,\lambda,\mu}(\Omega) \leq S_{\alpha,\beta,\lambda,\mu}(\Omega^r) = S_{\alpha,\beta,\lambda,\mu}(A^r) = S_{\alpha,\beta,\lambda,\mu}(A),$$

it follows that $S_{\alpha,\beta,\lambda,\mu}(\Omega) = S_{\alpha,\beta,\lambda,\mu}(A)$. $\square$

To search the results on general domains, we introduce the following marks:

$$\mathcal{A}^0(\Omega) := \{A : A \text{ is a cone and there exists some } r > 0 \text{ such that } \Omega_r \subseteq A\}$$

and

$$\mathcal{A}^\infty(\Omega) := \{A : A \text{ is a cone and there exists some } r > 0 \text{ such that } \Omega^r \subseteq A\}.$$

Notice that $\mathbb{R}^N \in \mathcal{A}^0(\Omega) \cap \mathcal{A}^\infty(\Omega)$, $\mathcal{A}^0(\Omega) \neq \emptyset, \mathcal{A}^\infty(\Omega) \neq \emptyset$. Then we can define

$$\tilde{S}^0_{\alpha,\beta,\lambda,\mu}(\Omega) := \sup\{S_{\alpha,\beta,\lambda,\mu}(A) : A \in \mathcal{A}^0(\Omega)\}$$

and

$$\tilde{S}^\infty_{\alpha,\beta,\lambda,\mu}(\Omega) := \sup\{S_{\alpha,\beta,\lambda,\mu}(A) : A \in \mathcal{A}^\infty(\Omega)\}.$$

Lemma 7.15.

$$\tilde{S}^0_{\alpha,\beta,\lambda,\mu}(\Omega) \leq S^0_{\alpha,\beta,\lambda,\mu}(\Omega), \quad \tilde{S}^\infty_{\alpha,\beta,\lambda,\mu}(\Omega) \leq S^\infty_{\alpha,\beta,\lambda,\mu}(\Omega).$$

Proof. We only prove $\tilde{S}^0_{\alpha,\beta,\lambda,\mu}(\Omega) \leq S^0_{\alpha,\beta,\lambda,\mu}(\Omega)$. For any $\varepsilon > 0$, there exists some $A \in \mathcal{A}^0(\Omega)$ such that

$$\tilde{S}^0_{\alpha,\beta,\lambda,\mu}(\Omega) - \varepsilon < S_{\alpha,\beta,\lambda,\mu}(A). \tag{7.129}$$

By the definition of $\mathcal{A}^0(\Omega)$, there exists some $r > 0$ such that $\Omega_r \subset A$. Then by the monotonicity property, we have

$$S_{\alpha,\beta,\lambda,\mu}(A) \leq S_{\alpha,\beta,\lambda,\mu}(\Omega_r) \leq S^0_{\alpha,\beta,\lambda,\mu}(\Omega). \tag{7.130}$$

By (7.129), (7.130) and the arbitrariness of ε, we obtain that $\tilde{S}^0_{\alpha,\beta,\lambda,\mu}(\Omega) \leq S^0_{\alpha,\beta,\lambda,\mu}(\Omega)$. $\square$

Remark 7.12. *By Corollary 7.4, if we can prove that*

$$S_{\alpha,\beta,\lambda,\mu}(\Omega) < \min\{S^0_{\alpha,\beta,\lambda,\mu}(\Omega), S^\infty_{\alpha,\beta,\lambda,\mu}(\Omega)\},$$

then $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ is attained. Since a lot of properties about cones have been studied in Section 7 (subsections 7.1-7.6), Lemma 7.15 supplies a useful way to compute $\min\{S^0_{\alpha,\beta,\lambda,\mu}(\Omega), S^\infty_{\alpha,\beta,\lambda,\mu}(\Omega)\}$. Here, we prefer to give some examples.

Example 1: If Ω is bounded with $0 \notin \bar{\Omega}$, then by $S_{\alpha,\beta,\lambda,\mu}(\emptyset) = +\infty$, we see that $S^0_{\alpha,\beta,\lambda,\mu}(\Omega) = S^\infty_{\alpha,\beta,\lambda,\mu}(\Omega) = +\infty$. Hence,

$$S_{\alpha,\beta,\lambda,\mu}(\Omega) < \min\{S^0_{\alpha,\beta,\lambda,\mu}(\Omega), S^\infty_{\alpha,\beta,\lambda,\mu}(\Omega)\}$$

and $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ is attained which can also deduce by Theorem 7.1.

The following examples are also given by Caldiroli, Paolo and Musina, Roberta [6], when they study the case of $p = 2, b = 0, a \in (-1, 0)$. What interesting is that the similar results still hold for our case (a slight modification on Example 3).

Example 2: Assume $0 \in \Omega$ is a cusp point, i.e., there exists a unit versor ν such that $\forall \theta \in (0, \pi), \exists\, r_\theta > 0$ such that $\Omega_{r_\theta} \subseteq \Omega_\theta$. Notice that

$$S^0_{\alpha,\beta,\lambda,\mu}(\Omega) = S^0_{\alpha,\beta,\lambda,\mu}(\Omega_{r_\theta}) \geq S_{\alpha,\beta,\lambda,\mu}(\Omega_{r_\theta}).$$

On the other hand,

$$S_{\alpha,\beta,\lambda,\mu}(\Omega_{r_\theta}) \geq S_{\alpha,\beta,\lambda,\mu}(\Omega_\theta) \to +\infty \text{ as } \theta \to 0.$$

Hence, $S^0_{\alpha,\beta,\lambda,\mu}(\Omega) = +\infty$.

Example 3: Let $\Omega = \Lambda \times \mathbb{R}^{N-k}, 1 \leq k < N$, where Λ is an open bounded domain of $\mathbb{R}^k$. Then there exists some $r > 0$ such that $\Lambda \subset B^k_r$, the ball in $\mathbb{R}^k$ with radial r. Now, we let

$$A_n := \big\{(tx', tx'') \in \mathbb{R}^k \times \mathbb{R}^{N-k} : t > 0, x' \in B^k_r, |x''|_{N-k} \geq n\big\},$$

then it is easy to see that $\{A_n\}$ are cones such that $A_n \supseteq A_{n+1}, \forall\, n$ and $\bigcap\limits_{n=1}^{\infty} A_n \subset \{0\} \times \mathbb{R}^{N-k}$. Thus, $meas(\bigcap_{n=1}^{\infty} A_n) = 0$. By (ii) of Lemma 7.11, $\lim\limits_{n\to\infty} S_{\alpha,\beta,\lambda,\mu}(A_n) = +\infty$. Define $\tilde{\Omega} = B^k_r \times \mathbb{R}^{N-k}$, then it is easy to see that $\Omega \subset \tilde{\Omega}$. Moreover, for any n, there exists some $r_n > \sqrt{r^2 + n^2} > 0$ such that $\tilde{\Omega}^{r_n} \subset A_n$, where $\tilde{\Omega}^{r_n}$ is defined by (7.120). Indeed, for any $x = (x_1, x_2) \in \tilde{\Omega} \backslash A_n$, we have $|x_1|_k < r$ and $\frac{|x_2|_{N-k}}{|x_1|_k} \leq \frac{n}{r}$, thus $|x_2|_{N-k} \leq n$. Then it follows that $|x|_N \leq \sqrt{r^2 + n^2}$. Hence, $\Omega^{r_n} \subset \tilde{\Omega}^{r_n} \subset A_n$. Then by the monotonicity property we have $S_{\alpha,\beta,\lambda,\mu}(\Omega^{r_n}) \geq S_{\alpha,\beta,\lambda,\mu}(\tilde{\Omega}^{r_n}) \geq S_{\alpha,\beta,\lambda,\mu}(A_n) \to +\infty$ as $n \to +\infty$. Hence $S^{+\infty}_{\alpha,\beta,\lambda,\mu}(\Omega) = \infty$.

Lemma 7.16. *Assume that* $s \in (0, 2), \kappa > 0, \alpha > 1, \beta > 1, \alpha + \beta = 2^*(s)$, *and* $\Omega = \Lambda \times \mathbb{R}^{N-k}, 1 \leq k < N$, *where* Λ *is an open bounded domain of* $\mathbb{R}^k$ *with* $0 \notin \bar{\Lambda}$. *Then* $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ *is attained.*

Proof. By Example 3, we see that $S^{\infty}_{\alpha,\beta,\lambda,\mu}(\Omega) = +\infty$. By $0 \notin \bar{\Lambda}$, we have $S^0_{\alpha,\beta,\lambda,\mu}(\Omega) = +\infty$. Then by Corollary 7.4, we obtain the conclusion. ☐

Based on the result of Lemma 7.14 and the maximum principle, we can obtain the following interesting results.

Corollary 7.6. *Assume that* $N \geq 3, s \in (0, 2), \kappa > 0, \alpha > 1, \beta > 1, \alpha + \beta = 2^*(s)$, *and let* Ω *be a general open domain of* $\mathbb{R}^N$.

(i) If $0 \in \Omega$, *then* $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ *is not attained unless* $\Omega = \mathbb{R}^N$*;*

(ii) If Ω is an exterior domain, then $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ is not attained unless $\Omega = \mathbb{R}^N$;

(iii) If $\Omega = A \cup U$, where U is an open bounded set with $0 \notin \bar{U}$ and A is a cone of $\mathbb{R}^N$, then

$$S_{\alpha,\beta,\lambda,\mu}(\Omega) < S_{\alpha,\beta,\lambda,\mu}(A) = S^0_{\alpha,\beta,\lambda,\mu}(\Omega) = S^\infty_{\alpha,\beta,\lambda,\mu}(\Omega)$$

and $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ is attained.

Proof. For the case of (i) and (ii), by Lemma 7.14, we see that

$$S_{\alpha,\beta,\lambda,\mu}(\Omega) = S_{\alpha,\beta,\lambda,\mu}(\mathbb{R}^N).$$

Then by the maximum principle, it is easy to see that $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ is not attained unless $\Omega = \mathbb{R}^N$. For the case of (iii), since U is bounded and $0 \notin \bar{U}$, there exists $r_1, r_2 > 0$ such that $\Omega_{r_1} = A_{r_1}, \Omega^{r_2} = A^{r_2}$. Hence $S^0_{\alpha,\beta,\lambda,\mu}(\Omega) = S^\infty_{\alpha,\beta,\lambda,\mu}(\Omega) = S_{\alpha,\beta,\lambda,\mu}(A)$. If $S_{\alpha,\beta,\lambda,\mu}(\Omega) < \min\{S^0_{\alpha,\beta,\lambda,\mu}(\Omega), \quad S^\infty_{\alpha,\beta,\lambda,\mu}(\Omega)\}$, then by Corollary 7.4, $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ is attained and the proof is completed. If not, by Corollary 7.4 again, $S_{\alpha,\beta,\lambda,\mu}(\Omega) = \min\{S^0_{\alpha,\beta,\lambda,\mu}(\Omega), S^\infty_{\alpha,\beta,\lambda,\mu}(\Omega)\}$, Hence, $S_{\alpha,\beta,\lambda,\mu}(\Omega) = S_{\alpha,\beta,\lambda,\mu}(A)$. By maximum principle again, $S_{\alpha,\beta,\lambda,\mu}(A)$ is not attained, a contradiction with Theorem 7.1. ☐

Theorem 7.9. *Assume that $N \geq 3, s \in (0,2), \kappa > 0, \alpha > 1, \beta > 1, \alpha + \beta = 2^*(s)$ and $\Omega \subset \mathbb{R}^N$ is an open bounded domain. If $0 \in \Omega$, then $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ is not attained.*

Proof. Since Ω is bounded, there exists some $r > 0$ such that $\Omega \subset B_r(0)$. When $0 \in \Omega$, by Lemma 7.14, we have

$$S_{\alpha,\beta,\lambda,\mu}(\Omega) = S_{\alpha,\beta,\lambda,\mu}(B_r(0)) = S_{\alpha,\beta,\lambda,\mu}(\mathbb{R}^N).$$

By way of negation, assume that $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ is attained and let $(u, v) \in D^{1,2}_0(\Omega) \times D^{1,2}_0(\Omega)$ be an extremal function. We may assume that $u \geq 0, v \geq 0$. By extending u and v outside Ω by 0, then we see that (u, v) is also an extremal function of $S_{\alpha,\beta,\lambda,\mu}(B_r(0)$. Hence, $(u, v) \neq (0, 0)$ is a nonnegative weak solution to the following problem:

$$\begin{cases} -\Delta u = S_{\alpha,\beta,a,b}(B_r(0))\Big(\lambda \frac{1}{|x|^s}|u|^{2^*(s)-2}u + \kappa\alpha \frac{1}{|x|^s}|u|^{\alpha-2}u|v|^\beta\Big) & \text{in } B_r(0),\\ -\Delta v = S_{\alpha,\beta,a,b}(B_r(0))\Big(\mu \frac{1}{|x|^s}|v|^{2^*(s)-2}v + \kappa\beta \frac{1}{|x|^s}|u|^{\alpha}|v|^{\beta-2}v\Big) & \text{in } B_r(0),\\ u \geq 0, v \geq 0, (u, v) \in \mathscr{D} := D^{1,2}_0(B_r(0)) \times D^{1,2}_0(B_r(0)). \end{cases} \tag{7.131}$$

On the other hand, since $B_r(0)$ is a star-shaped domain, by Proposition 7.3 (see the formula (7.28)), problem(7.131) has no nontrivial solution even semi-trivial solution, a contradiction. Hence, we know that $S_{\alpha,\beta,\lambda,\mu}(\Omega)$ is not attained. ☐

Corollary 7.7. *Assume that $N \geq 3, s \in (0,2), \kappa > 0, \alpha > 1, \beta > 1, \alpha + \beta = 2^*(s)$, if there exist some $r_1, r_2 > 0$ and $\theta \in (0, \pi]$ such that*

$$\big(\Omega_\theta \cap B_{r_1}(0)\big) \subsetneq \Omega \subsetneq \big(\Omega_\theta \cap B_{r_2}(0)\big),$$

then $S_{\alpha,\beta,\lambda,\mu}(\Omega) = S_{\alpha,\beta,\lambda,\mu}(\Omega_\theta)$ and is not attained.

Proof. The proof is similar to that of Theorem 7.9, we omit the details. ☐

8 The case of $s_1 \neq s_2 \in (0,2)$

In this section, we study the case of $s_1 \neq s_2 \in (0,2)$. By constructing a new approximation, the existence of positive ground state solution to the system (1.1) will be obtained, including the regularity and decay estimation.

Assume Ω is cone. Define

$$U_\lambda := \Big(\frac{\mu_{s_1}(\Omega)}{\lambda}\Big)^{\frac{1}{2^*(s_1)-2}} U,$$

where U is a ground state solution to the following problem:

$$\begin{cases} -\Delta u = \mu_{s_1}(\Omega)\frac{u^{2^*(s_1)-1}}{|x|^{s_1}} \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{8.1}$$

Set

$$\eta_{1,0} := \inf_{v\in\Xi_0} \|v\|^2, \qquad \eta_{2,0} := \inf_{u\in\Theta_0} \|u\|^2, \tag{8.2}$$

where

$$\Xi_0 := \Big\{ v \in D_0^{1,2}(\Omega) : \int_\Omega \frac{1}{|x|^{s_2}} |U_\lambda|^{2^*(s_2)-2}|v|^2 dx = 1 \Big\}, \tag{8.3}$$

$$\Theta_0 := \Big\{ u \in D_0^{1,2}(\Omega) : \int_\Omega \frac{1}{|x|^{s_2}} |U_\mu|^{2^*(s_2)-2}|u|^2 dx = 1 \Big\}. \tag{8.4}$$

The corresponding energy functional of the problem (1.1) is defined as

$$\Phi_0(u,v) = \frac{1}{2}a(u,v) - \frac{1}{2^*(s_1)}b(u,v) - \kappa c(u,v) \tag{8.5}$$

for all $(u,v) \in \mathscr{D}$, where

$$\begin{cases} a(u,v) := \|(u,v)\|^2_{\mathscr{D}}, \\ b(u,v) := \lambda \int_\Omega \frac{|u|^{2^*(s_1)}}{|x|^{s_1}}dx + \mu \int_\Omega \frac{|v|^{2^*(s_1)}}{|x|^{s_1}}dx, \\ c(u,v) := \int_\Omega \frac{1}{|x|^{s_2}}|u|^\alpha |v|^\beta dx. \end{cases} \tag{8.6}$$

Here comes our main result in this section:

Theorem 8.1. *Assume that* $s_1, s_2 \in (0,2), \lambda, \mu \in (0,+\infty), \kappa > 0, \alpha > 1, \beta > 1, \alpha + \beta = 2^*(s_2)$. *Suppose that one of the following holds:*

(a_1) $\lambda > \mu$ *and either* $1 < \beta < 2$ *or* $\begin{cases} \beta = 2 \\ \kappa > \frac{\eta_{1,0}}{2^*(s_2)} \end{cases}$;

(a_2) $\lambda = \mu$ *and either* $\min\{\alpha,\beta\} < 2$ *or* $\begin{cases} \min\{\alpha,\beta\} = 2, \\ \kappa > \frac{\eta_{1,0}}{2^*(s_2)} = \frac{\eta_{2,0}}{2^*(s_2)} \end{cases}$;

(a_3) $\lambda < \mu$ *and either* $1 < \alpha < 2$ *or* $\begin{cases} \alpha = 2 \\ \kappa > \frac{\eta_{2,0}}{2^*(s_2)} \end{cases}$.

Then problem (1.1) *possesses a positive ground state solution* (u_0, v_0) *such that*

$$\Phi_0(u_0, v_0) < \Big[\frac{1}{2} - \frac{1}{2^*(s_1)}\Big]\big(\mu_{s_1}(\Omega)\big)^{\frac{2^*(s_1)}{2^*(s_1)-2}}\big(\max\{\lambda, \mu\}\big)^{-\frac{2}{2^*(s_1)-2}}.$$

Moreover, we have the following regularity and decay propositions:

(b_1) *if* $0 < \max\{s_1, s_2\} < \frac{N+2}{N}$*, then* $u_0, v_0 \in C^2(\overline{\Omega})$*;*

(b_2) *if* $\max\{s_1, s_2\} = \frac{N+2}{N}$*, then* $u_0, v_0 \in C^{1,\gamma}(\Omega)$ *for all* $0 < \gamma < 1$*;*

(b_3) *if* $\max\{s_1, s_2\} > \frac{N+2}{N}$*, then* $u_0, v_0 \in C^{1,\gamma}(\Omega)$ *for all* $0 < \gamma < \frac{N(2-\max\{s_1,s_2\})}{N-2}$.

When Ω *is a cone with* $0 \in \partial\Omega$ *(e.g.,* $\Omega = \mathbb{R}^N_+$*), then there exists a constant* C *such that*

$$|u_0(x)|, |v_0(x)| \le C(1 + |x|^{-(N-1)}), \qquad |\nabla u_0(x)|, |\nabla v_0(x)| \le C|x|^{-N}.$$

When $\Omega = \mathbb{R}^N$*,*

$$|u_0(x)|, |v_0(x)| \le C(1 + |x|^{-N}), \qquad |\nabla u_0(x)|, |\nabla v_0(x)| \le C|x|^{-N-1}$$

In particular, if $\Omega = \mathbb{R}^N_+$*, then* $(u_0(x), v_0(x))$ *is axially symmetric with respect to the* x_N*-axis, i.e.,*

$$\big(u_0(x), v_0(x)\big) = \big(u_0(x', x_N), v_0(x', x_N)\big) = \big(u_0(|x'|, x_N), v_0(|x'|, x_N)\big).$$

Remark 8.1. *The regularity, symmetry results and the decay estimation we have established in Section 3 of the present paper. Therefore, in the current section we only need to focus on the existence of the positive ground state solution.*

8.1 Approximation

When $s_1 \neq s_2$, the nonlinearities are not homogeneous any more which make the problem much tough. Here we have to choose a different approximation to the original problem in the same domain, i.e., we consider the following problem:

$$\begin{cases} -\Delta u - \lambda \frac{1}{|x|^{s_1}}|u|^{2^*(s_1)-2}u = \kappa\alpha a_\varepsilon(x)|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ -\Delta v - \mu \frac{1}{|x|^{s_1}}|v|^{2^*(s_1)-2}v = \kappa\beta a_\varepsilon(x)|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ \kappa > 0, (u, v) \in \mathscr{D} := D_0^{1,2}(\Omega) \times D_0^{1,2}(\Omega), \end{cases} \tag{8.7}$$

where

$$a_\varepsilon(x) := \begin{cases} \frac{1}{|x|^{s_2-\varepsilon}} & \text{for } |x| < 1 \\ \frac{1}{|x|^{s_2+\varepsilon}} & \text{for } |x| \ge 1 \end{cases} \qquad \text{for } \varepsilon \in [0, s_2). \tag{8.8}$$

Under some proper assumptions on $\alpha, \beta, \lambda, \mu$ and $\kappa > 0$, we shall prove the existence of the positive ground state solution $(u_\varepsilon, v_\varepsilon)$ to (8.7) with a well-dominated energy (see Theorem 8.2 below). Finally, we can approach an existence result of (1.1).

The corresponding energy functional of problem (8.7) is defined as

$$\Phi_\varepsilon(u,v) = \frac{1}{2}a(u,v) - \frac{1}{2^*(s_1)}b(u,v) - \kappa c_\varepsilon(u,v) \tag{8.9}$$

for all $(u,v) \in \mathscr{D}$, where $a(u,v)$ and $b(u,v)$ are defined in (8.6) and

$$c_\varepsilon(u,v) := \int_\Omega a_\varepsilon(x)|u|^\alpha |v|^\beta dx, \tag{8.10}$$

which is decreasing by ε. Consider the corresponding Nehari manifold

$$\mathcal{N}_\varepsilon := \{(u,v) \in \mathscr{D}\backslash(0,0) : \ J_\varepsilon(u,v) = 0\}$$

where

$$J_\varepsilon(u,v) := \langle \Phi'_\varepsilon(u,v),(u,v)\rangle = a(u,v) - b(u,v) - \kappa(\alpha+\beta)c_\varepsilon(u,v). \tag{8.11}$$

Lemma 8.1. *Assume* $s_1, s_2 \in (0,2), \lambda, \mu \in (0,+\infty), \kappa > 0, \alpha > 1, \beta > 1$ *and* $\alpha + \beta = 2^*(s_2)$. *Let* $\varepsilon \in [0, s_2)$, *then for any* $(u,v) \in \mathscr{D}\backslash\{(0,0)\}$, *there exists a unique* $t = t_{(\varepsilon,u,v)} > 0$ *such that* $(tu, tv) \in \mathcal{N}_\varepsilon$. *Moreover,* $\mathcal{N}_\varepsilon$ *is closed and bounded away from* $(0,0)$. *Further,* $t = t_{(\varepsilon,u,v)}$ *is increasing by* ε.

Proof. The existence and uniqueness of $t = t_{(\varepsilon,u,v)}$ and that $\mathcal{N}_\varepsilon$ is closed and bounded away from 0, we refer to Lemma 4.1. Now, we prove that $t = t_{(\varepsilon,u,v)}$ is increasing by ε. Assume that $0 \le \varepsilon_1 < \varepsilon_2 < s_2$, then we see that there exists a unique t_1 and t_2 such that

$$J_{\varepsilon_1}(t_1 u, t_1 v) = J_{\varepsilon_2}(t_2 u, t_2 v) = 0. \tag{8.12}$$

Recalling that $c_\varepsilon(u,v)$ is decreasing by ε, we see that $J_\varepsilon(u,v)$ is increasing by ε. Hence,

$$J_{\varepsilon_2}(t_1 u, t_1 v) \ge J_{\varepsilon_1}(t_1 u, t_1 v) = 0. \tag{8.13}$$

If $J_{\varepsilon_2}(t_1 u, t_1 v) = 0$, by the uniqueness, we obtain that $t_2 = t_1$. If $J_{\varepsilon_2}(t_1 u, t_1 v) > 0$, noting that $J_{\varepsilon_2}(tu, tv) \to -\infty$ as $t \to +\infty$, there exists some $t_* > t_1$ such that $J_{\varepsilon_2}(t_* u, t_* v) = 0$. Then by the uniqueness again, we see that $t_2 = t_* > t_1$. Hence, we always have $t_2 \ge t_1$ and we note that $t_2 > t_1$ when $uv \not\equiv 0$. $\square$

Define

$$c_\varepsilon := \inf_{(u,v)\in\mathcal{N}_\varepsilon} \Phi_\varepsilon(u,v), \qquad \delta_\varepsilon := \inf_{(u,v)\in\mathcal{N}_\varepsilon} \sqrt{\|u\|^2 + \|v\|^2}. \tag{8.14}$$

We have the following results:

Lemma 8.2. δ_ε *is increasing by* $\varepsilon \in [0, s_2)$, *i.e.,* $\delta_0 \le \delta_{\varepsilon_1} \le \delta_{\varepsilon_2}$ *provided* $0 \le \varepsilon_1 < \varepsilon_2 < s_2$.

Proof. For any $(u,v) \ne (0,0)$, set $\phi = \frac{u}{\sqrt{\|u\|^2+\|v\|^2}}, \psi = \frac{v}{\sqrt{\|u\|^2+\|v\|^2}}$. By Lemma 8.1, there exists $0 < t_1 \le t_2$ such that $(t_1\phi, t_1\psi) \in \mathcal{N}_{\varepsilon_1}$ and $(t_2\phi, t_2\psi) \in \mathcal{N}_{\varepsilon_2}$. Hence, we obtain that δ_ε is increasing by $\varepsilon \in [0, s_2)$. $\square$

Remark 8.2. *Set* $s_{max} := \max\{s_1, s_2\}$*, it is easy to prove that for any* $(u,v) \in \mathcal{N}_\varepsilon$*, we have*

$$\Phi_\varepsilon(u,v) \geq \Big(\frac{1}{2} - \frac{1}{2^*(s_{max})}\Big)\big(\|u\|^2 + \|v\|^2\big) \tag{8.15}$$

and it follows that

$$c_\varepsilon \geq \Big(\frac{1}{2} - \frac{1}{2^*(s_{max})}\Big)\delta_\varepsilon^2. \tag{8.16}$$

Lemma 8.3. c_ε *is increasing by* ε *in* $[0, s_2)$.

Proof. Let $(\phi,\psi) \neq (0,0)$ be fixed. By Lemma 8.1, for any $\varepsilon \in [0, s_2)$, there exists a unique $t_\varepsilon > 0$ such that $t_\varepsilon(\phi,\psi) \in \mathcal{N}_\varepsilon$. In fact, t_ε is implicitly defined by the equation

$$a(\phi,\psi) - b(\phi,\psi)t_\varepsilon^{2^*(s_1)-2} - 2^*(s_2)\kappa c_\varepsilon(\phi,\psi)t_\varepsilon^{2^*(s_2)-2} = 0. \tag{8.17}$$

It then follows that

$$\begin{aligned}&\Phi_\varepsilon\big(t(\varepsilon)\phi, t(\varepsilon)\psi\big)\\ =& \Big[\frac{1}{2} - \frac{1}{2^*(s_2)}\Big]a(\phi,\psi)[t(\varepsilon)]^2 + \Big[\frac{1}{2^*(s_2)} - \frac{1}{2^*(s_1)}\Big]b(\phi,\psi)[t(\varepsilon)]^{2^*(s_1)}.\end{aligned} \tag{8.18}$$

Case 1: $s_2 > s_1$**.** For this case, we see that $\frac{1}{2^*(s_2)} - \frac{1}{2^*(s_1)} > 0$. Noting that $a(\phi,\psi) > 0, b(\phi,\psi) > 0$ and Lemma 8.1, we obtain that

$$\Phi_\varepsilon\big(t(\varepsilon)\phi, t(\varepsilon)\psi\big) \text{ is increasing by } \varepsilon \text{ in } [0, s_2). \tag{8.19}$$

Hence, we get that c_ε is increasing by ε in $[0, s_2)$.

Case 2: $s_2 < s_1$**.** By the Implicit Function Theorem, we see that $t(\varepsilon) \in C^1(\mathbb{R})$ and $\frac{d}{d\varepsilon}t(\varepsilon) \geq 0$ by Lemma 8.1. Hence,

$$\begin{aligned}&\frac{d}{d\varepsilon}\Phi_\varepsilon\big(t(\varepsilon)\phi, t(\varepsilon)\psi\big)\\ =&2[\frac{1}{2} - \frac{1}{2^*(s_2)}]a(\phi,\psi)t(\varepsilon)t'(\varepsilon) + 2^*(s_1)[\frac{1}{2^*(s_2)} - \frac{1}{2^*(s_1)}]b(\phi,\psi)[t(\varepsilon)]^{2^*(s_1)-1}t'(\varepsilon)\\ =&\frac{t'(\varepsilon)}{t(\varepsilon)}\Big[[1 - \frac{2}{2^*(s_2)}]a(\phi,\psi)[t(\varepsilon)]^2 + [\frac{2^*(s_1)}{2^*(s_2)} - 1]b(\phi,\psi)[t(\varepsilon)]^{2^*(s_1)}\Big]\\ =&\frac{t'(\varepsilon)}{t(\varepsilon)}\Big[[1 - \frac{2}{2^*(s_2)}]a(\phi,\psi)[t(\varepsilon)]^2\\ &\quad + [\frac{2^*(s_1)}{2^*(s_2)} - 1]\big[a(\phi,\psi)[t(\varepsilon)]^2 - 2^*(s_2)\kappa c_\varepsilon(\phi,\psi)[t(\varepsilon)]^{2^*(s_2)}\big]\Big]\\ =&\frac{t'(\varepsilon)}{t(\varepsilon)}\Big[\frac{2^*(s_1)-2}{2^*(s_2)}a(\phi,\psi)[t(\varepsilon)]^2 + [2^*(s_2) - 2^*(s_1)]\kappa c_\varepsilon(\phi,\psi)[t(\varepsilon)]^{2^*(s_2)}\Big]\\ \geq&0.\end{aligned} \tag{8.20}$$

Hence, we also obtain the conclusion of (8.19) for the case of $s_2 < s_1$ and the proof is completed. ◻

8.2 Estimation on the least energy of the approximation

Recall $U_\lambda := \left(\frac{\mu_{s_1}(\Omega)}{\lambda}\right)^{\frac{1}{2^*(s_1)-2}} U$, where U is a ground state solution of the following problem:

$$\begin{cases} -\Delta u = \mu_{s_1}(\Omega)\frac{u^{2^*(s_1)-1}}{|x|^{s_1}} \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{8.21}$$

Define the function

$$\Psi_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2^*(s_1)}\int_\Omega \frac{|u|^{2^*(s_1)}}{|x|^{s_1}} dx. \tag{8.22}$$

Then

$$m_\lambda = \Psi_\lambda(U_\lambda) = [\frac{1}{2} - \frac{1}{2^*(s_1)}]\big(\mu_{s_1}(\Omega)\big)^{\frac{2^*(s_1)}{2^*(s_1)-2}} \lambda^{-\frac{2}{2^*(s_1)-2}} \tag{8.23}$$

is the least energy.

Remark 8.3. *Evidently, for any $\varepsilon \in [0, s_2)$, we have that $c_\varepsilon \leq m_\lambda$ and $c_\varepsilon \leq m_\mu$. Hence,*

$$c_\varepsilon \leq [\frac{1}{2} - \frac{1}{2^*(s_1)}]\big(\mu_{s_1}(\Omega)\big)^{\frac{2^*(s_1)}{2^*(s_1)-2}} \big(\max\{\lambda, \mu\}\big)^{-\frac{2}{2^*(s_1)-2}}. \tag{8.24}$$

□

Define

$$\eta_{1,\varepsilon} := \inf_{v \in \Xi_\varepsilon} \|v\|^2 \tag{8.25}$$

where

$$\Xi_\varepsilon := \{v \in D_0^{1,2}(\Omega) : \int_\Omega a_\varepsilon(x)|U_\lambda|^{2^*(s_2)-2}|v|^2 dx = 1\}. \tag{8.26}$$

Since $a_\varepsilon(x)$ is decreasing by ε, it is easy to see that $\eta_{1,\varepsilon}$ is increasing by ε.

Lemma 8.4. *Assume $s_1, s_2 \in (0,2), \lambda, \mu \in (0, +\infty), \kappa > 0, \alpha > 1, \beta > 1$ and $\alpha + \beta = 2^*(s_2)$. Let $\varepsilon \in [0, s_2)$.*

(1) *If $\beta < 2$, then $c_\varepsilon < m_\lambda$.*

(2) *If $\beta > 2$, then $(U_\lambda, 0)$ is a local minimum point of Φ_ε in $\mathcal{N}_\varepsilon$.*

(3) *If $\beta = 2$ and $\kappa > \frac{\eta_{1,\varepsilon}}{2^*(s_2)}$, then $c_\varepsilon < m_\lambda$.*

(4) *If $\beta = 2$ and $0 < \kappa < \frac{\eta_{1,\varepsilon}}{2^*(s_2)}$, then $(U_\lambda, 0)$ is a local minimum point of Φ_ε in $\mathcal{N}_\varepsilon$.*

Proof. The proofs are similar to those in Section 6.2. □

Lemma 8.5. *$\eta_{1,\varepsilon}$ is continuous with respect to $\varepsilon \in [0, s_2)$.*

Proof. For any $\varepsilon_0 \in [0, s_2)$, we shall prove that $\eta_{1,\varepsilon}$ is continuous at $\varepsilon = \varepsilon_0$. Apply the argument of Lemma 6.5, there exists some $0 < v_0 \in D_0^{1,2}(\Omega)$ such that

$$\|v_0\|^2 = \eta_{1,\varepsilon_0} \text{ and } \int_\Omega a_{\varepsilon_0}(x)|U_\lambda|^{2^*(s_2)-2} v_0^2 dx = 1. \tag{8.27}$$

Take a sequence $\{\varepsilon_n\} \subset [0, s_2)$ with $\varepsilon_n \downarrow \varepsilon_0$ as $n \to +\infty$. Recall that $\eta_{1,\varepsilon}$ is increasing by ε, then $\lim_{n\to+\infty} \eta_{1,\varepsilon_n}$ exists and satisfies

$$\lim_{n\to+\infty} \eta_{1,\varepsilon_n} \geq \eta_{1,\varepsilon_0}. \tag{8.28}$$

On the other hand, since $a_{\varepsilon_n}(x) \to a_0(x)$ a.e. in Ω, recalling the decay property of U_λ (see [13, Theorem 1.2], [15, Lemma 2.1], [19, Lemma 2.6]), it is easy to prove that

$$\lim_{n\to\infty} \int_\Omega a_{\varepsilon_n}(x)|U_\lambda|^{2^*(s_2)-2} v_0^2 dx = \int_\Omega a_{\varepsilon_0}(x)|U_\lambda|^{2^*(s_2)-2} v_0^2 dx = 1. \tag{8.29}$$

Hence,

$$\lim_{n\to+\infty} \frac{\|v_0\|^2}{\int_\Omega a_{\varepsilon_n}(x)|U_\lambda|^{2^*(s_2)-2} v_0^2 dx} = \eta_{1,\varepsilon_0}. \tag{8.30}$$

Then by the definition of $\eta_{1,\varepsilon}$, we see that

$$\lim_{n\to+\infty} \eta_{1,\varepsilon_n} \leq \lim_{n\to+\infty} \frac{\|v_0\|^2}{\int_\Omega a_{\varepsilon_n}(x)|U_\lambda|^{2^*(s_2)-2} v_0^2 dx} = \eta_{1,\varepsilon_0}. \tag{8.31}$$

By (8.28) and (8.31), we obtain that $\eta_{1,\varepsilon}$ is right-continuous.

Secondly, we take a sequence $\{\varepsilon_n\} \subset [0, s_2)$ such that $\varepsilon_n \uparrow \varepsilon_0$ as $n \to +\infty$. By Lemma Lemma 6.5 again, we may assume that $\{v_n\} \subset D_0^{1,2}(\Omega)$ such that

$$\|v_n\|^2 = \eta_{1,\varepsilon_n} \text{ and } \int_\Omega a_{\varepsilon_n}(x)|U_\lambda|^{2^*(s_2)-2} v_n^2 dx \equiv 1. \tag{8.32}$$

Up to a subsequence, we may assume that $v_n \rightharpoonup v_0$ in $D_0^{1,2}(\Omega)$ and $v_n \to v_0$ a.e. in Ω. Similarly, we can prove that

$$\int_\Omega a_{\varepsilon_0}(x)|U_\lambda|^{2^*(s_2)-2} v_0^2 dx = \lim_{n\to+\infty} \int_\Omega a_{\varepsilon_n}(x)|U_\lambda|^{2^*(s_2)-2} v_n^2 dx = 1. \tag{8.33}$$

It follows that

$$\|v_0\|^2 \leq \liminf_{n\to+\infty} \|v_n\|^2 = \lim_{n\to+\infty} \eta_{1,\varepsilon_n}. \tag{8.34}$$

Therefore,

$$\eta_{1,\varepsilon_0} \leq \frac{\|v_0\|^2}{\int_\Omega a_{\varepsilon_0}(x)|U_\lambda|^{2^*(s_2)-2} v_0^2 dx} \leq \lim_{n\to+\infty} \eta_{1,\varepsilon_n}. \tag{8.35}$$

On the other hand, by the monotonicity, we can obtain that reverse inequality. Hence,

$$\eta_{1,\varepsilon_0} \leq \frac{\|v_0\|^2}{\int_\Omega a_{\varepsilon_0}(x)|U_\lambda|^{2^*(s_2)-2} v_0^2 dx} = \lim_{n\to+\infty} \eta_{1,\varepsilon_n},$$

i.e., $\eta_{1,\varepsilon}$ is left-continuous. The proof is completed. $\square$

Similarly, we define

$$\eta_{2,\varepsilon} := \inf_{u\in\Theta_\varepsilon} \|u\|^2 \tag{8.36}$$

where

$$\Theta_\varepsilon := \Big\{ u \in D_0^{1,2}(\Omega) : \int_\Omega a_\varepsilon(x)|U_\mu|^{2^*(s_2)-2}|u|^2 dx = 1 \Big\}. \tag{8.37}$$

We also have that $\eta_{2,\varepsilon}$ is increasing by $\varepsilon \in [0, s_2)$ and continuous. Furthermore, we can propose the following results without proof.

Lemma 8.6. *Assume* $s_1, s_2 \in (0,2), \lambda, \mu \in (0,+\infty), \kappa > 0, \alpha > 1, \beta > 1$ *and* $\alpha + \beta = 2^*(s_2)$. *Let* $\varepsilon \in [0, s_2)$.

(1) *If* $\alpha < 2$*, then* $c_\varepsilon < m_\mu$.

(2) *If* $\alpha > 2$*, then* $(0, U_\mu)$ *is a local minimum point of* Φ_ε *in* $\mathcal{N}_\varepsilon$.

(3) *If* $\alpha = 2, \kappa > \frac{\eta_{2,\varepsilon}}{2^*(s_2)}$*, then* $c_\varepsilon < m_\mu$.

(4) *If* $\alpha = 2, 0 < \kappa < \frac{\eta_{2,\varepsilon}}{2^*(s_2)}$*, then* $(0, U_\mu)$ *is a local minimum point of* Φ_ε *in* $\mathcal{N}_\varepsilon$.

Now we can obtain the following estimation on c_ε:

Lemma 8.7. *Assume* $s_1, s_2 \in (0,2), \lambda, \mu \in (0,+\infty), \kappa > 0, \alpha > 1, \beta > 1$ *and* $\alpha + \beta = 2^*(s_2)$. *Let* $\varepsilon \in [0, s_2)$*, then we have*

$$c_\varepsilon < \min\{m_\lambda, m_\mu\} = \left[\frac{1}{2} - \frac{1}{2^*(s_1)}\right] \Big(\mu_{s_1}(\Omega)\Big)^{\frac{2^*(s_1)}{2^*(s_1)-2}} \Big(\max\{\lambda,\mu\}\Big)^{-\frac{2}{2^*(s_1)-2}}$$

if one of the following holds:

(a) $\lambda > \mu$ *and either* $1 < \beta < 2$ *or* $\begin{cases} \beta = 2 \\ \kappa > \frac{\eta_{1,\varepsilon}}{2^*(s_2)} \end{cases}$ *;*

(b) $\lambda = \mu$ *and either* $\min\{\alpha,\beta\} < 2$ *or* $\begin{cases} \min\{\alpha,\beta\} = 2, \\ \kappa > \frac{\eta_{1,\varepsilon}}{2^*(s_2)} = \frac{\eta_{2,\varepsilon}}{2^*(s_2)} \end{cases}$ *;*

(c) $\lambda < \mu$ *and either* $1 < \alpha < 2$ *or* $\begin{cases} \alpha = 2 \\ \kappa > \frac{\eta_{2,\varepsilon}}{2^*(s_2)} \end{cases}$.

Proof. It is a direct conclusion following by Lemma 8.4 and Lemma 8.6. $\square$

8.3 Positive ground state to the approximation problem (8.7)

In this subsection, we assume that $\varepsilon \in (0, s_2)$ is fixed. Then we can obtain the following result.

Theorem 8.2. *Assume* $s_1, s_2 \in (0,2), \lambda, \mu \in (0,+\infty), \kappa > 0, \alpha > 1, \beta > 1$ *and* $\alpha+\beta = 2^*(s_2)$. *Then problem* (8.7) *possesses a positive ground state solution* $(\phi_\varepsilon, \psi_\varepsilon)$ *provided further one of the following conditions holds:*

(1) $\lambda > \mu$ *and either* $1 < \beta < 2$ *or* $\begin{cases} \beta = 2 \\ \kappa > \frac{\eta_{1,\varepsilon}}{2^*(s)} \end{cases}$ *;*

(2) $\lambda = \mu$ *and either* $\min\{\alpha,\beta\} < 2$ *or* $\begin{cases} \min\{\alpha,\beta\} = 2, \\ \kappa > \frac{\eta_{1,\varepsilon}}{2^*(s)} = \frac{\eta_{2,\varepsilon}}{2^*(s)} \end{cases}$ *;*

(3) $\lambda < \mu$ *and either* $1 < \alpha < 2$ *or* $\begin{cases} \alpha = 2 \\ \kappa > \frac{\eta_{2,\varepsilon}}{2^*(s)} \end{cases}$.

Proposition 8.1. *Assume that* $\varepsilon \in (0, s_2)$ *and* $\{(u_n, v_n)\}$ *is a bounded* $(PS)_c$ *sequence of* Φ_ε. *Up to a subsequence, we assume that* $(u_n, v_n) \rightharpoonup (\phi, \psi)$ *weakly in* $\mathscr{D}$. *Set* $\tilde{u}_n := u_n - \phi, \tilde{v}_n := v_n - \psi$, *then we have that*

$$\Psi'_\lambda(\tilde{u}_n) \to 0 \text{ and } \Psi'_\mu(\tilde{v}_n) \to 0 \text{ in } H^{-1}(\Omega), \tag{8.38}$$

where Ψ_λ *is defined in* (8.22).

Proof. Under the assumptions, we see that

$$\left\langle \Phi'_\varepsilon(u_n, v_n), (h, 0)\right\rangle = o(1)\|h\| \tag{8.39}$$

Since $(u_n, v_n) \rightharpoonup (\phi, \psi)$, it is easy to see that $\Phi'_\varepsilon(\phi, \psi) = 0$. Then we have

$$\left\langle \Phi'_\varepsilon(\phi, \psi), (h, 0)\right\rangle = 0. \tag{8.40}$$

By Lemma 7.5 and Hölder inequality, it is easy to see that

$$\int_\Omega a_\varepsilon(x)|u_n|^{\alpha-2}u_n|v_n|^\beta h dx - \int_\Omega a_\varepsilon(x)|\phi|^{\alpha-2}\phi|\psi|^\beta h dx = o(1)\|h\|. \tag{8.41}$$

It follows from (8.39),(8.40) and (8.41) that

$$\int_\Omega \nabla(u_n - \phi)\nabla h dx - \lambda \int_\Omega \Big(\frac{|u_n|^{2^*(s_1)-2}u_n}{|x|^{s_1}} - \frac{|\phi|^{2^*(s_1)-2}\phi}{|x|^{s_1}}\Big) h dx = o(1)\|h\|. \tag{8.42}$$

By [12, Lemma 3.3] or [8, Lemma 3.2], we see that

$$\frac{|u_n|^{2^*(s_1)-2}u_n}{|x|^{s_1}} - \frac{|u_n - \phi|^{2^*(s_1)-2}(u_n - \phi)}{|x|^{s_1}} \to \frac{|\phi|^{2^*(s_1)-2}\phi}{|x|^{s_1}} \text{ in } H^{-1}(\Omega). \tag{8.43}$$

Hence, by (8.42) and (8.43), we obtain that

$$\Psi'_\lambda(\tilde{u}_n) \to 0 \text{ in } H^{-1}(\Omega). \tag{8.44}$$

Apply the similar arguments, we can prove that $\Psi'_\mu(\tilde{v}_n) \to 0$ in $H^{-1}(\Omega)$. □

Corollary 8.1. *Under the assumptions of Proposition 8.1 and furthermore we assume that*

$$c < \min\{m_\lambda, m_\mu\} = \left[\frac{1}{2} - \frac{1}{2^*(s_1)}\right] \big(\mu_{s_1}(\Omega)\big)^{\frac{2^*(s_1)}{2^*(s_1)-2}} \big(\max\{\lambda,\mu\}\big)^{-\frac{2}{2^*(s_1)-2}}.$$

Then up to a subsequence, $(u_n, v_n) \to (\phi, \psi)$ *strongly in* $\mathscr{D}$ *and* (ϕ,ψ) *satisfies*

$$\Phi_\varepsilon(\phi,\psi) = c \text{ and } \Phi'_\varepsilon(\phi,\psi) = 0 \text{ in } \mathscr{D}^*.$$

Proof. We prove it by way of negation. Assume that $(u_n, v_n) \not\to (\phi,\psi)$, then at least one of the following holds:

(i) $u_n \not\to \phi$ in $D_0^{1,2}(\Omega)$;

(ii) $v_n \not\to \psi$ in $D_0^{1,2}(\Omega)$.

Without loss of generality, we assume (i). By Proposition 8.1, we see that $\Psi'_\lambda(\tilde{u}_n) \to 0$ in $H^{-1}(\Omega)$. Since $\tilde{u}_n = u_n - \phi \not\to 0$ in $D_0^{1,2}(\Omega)$, it is easy to see that

$$\liminf_{n\to+\infty} \Psi_\lambda(\tilde{u}_n) \geq m_\lambda. \tag{8.45}$$

On the other hand, by the Brézis-Lieb type lemma (see [12, Lemma 3.3]), we have

$$\Phi_\varepsilon(u_n, v_n) = \Phi_\varepsilon(\tilde{u}_n, \tilde{v}_n) + \Phi_\varepsilon(\phi,\psi) + o(1). \tag{8.46}$$

By Lemma 7.5 again, we see that

$$\Phi_\varepsilon(\tilde{u}_n, \tilde{v}_n) = \Psi_\lambda(\tilde{u}_n) + \Psi_\mu(\tilde{v}_n) + o(1). \tag{8.47}$$

Since $\Psi'_\mu(\tilde{v}_n) \to 0$ in $H^{-1}(\Omega)$, it is easy to prove that $\liminf\limits_{n\to+\infty} \Psi_\mu(\tilde{v}_n) \geq 0$. We also note that $\Phi_\varepsilon(\phi,\psi) \geq 0$. Then by (8.46), (8.47) and (8.45), we have

$$c = \lim_{n\to+\infty} \Phi_\varepsilon(u_n, v_n) \geq \lim_{n\to+\infty} \Psi_\lambda(\tilde{u}_n) \geq m_\lambda, \tag{8.48}$$

a contradiction. □

Proof of Theorem 8.2: Let $\{(u_n, v_n)\} \subset \mathcal{N}_\varepsilon$ be a minimizing sequence. Then it is easy to see that

$$\Phi_\varepsilon(u_n, v_n) \to c_\varepsilon \text{ and } \Phi'_\varepsilon\big|_{\mathcal{N}_\varepsilon}(u_n, v_n) \to 0 \text{ in } \mathscr{D}^*.$$

It is standard to prove that (u_n, v_n) is bounded in $\mathscr{D}$ and is also a $(PS)_{c_\varepsilon}$ sequence of Φ_ε. By Lemma 8.7, we have

$$c_\varepsilon < \min\{m_\lambda, m_\mu\} = [\frac{1}{2} - \frac{1}{2^*(s_1)}] \big(\mu_{s_1}(\Omega)\big)^{\frac{2^*(s_1)}{2^*(s_1)-2}} \big(\max\{\lambda,\mu\}\big)^{-\frac{2}{2^*(s_1)-2}}. \tag{8.49}$$

Hence, by Corollary 8.1, there exists some $(\phi, \psi) \in \mathscr{D}$ and up to a subsequence, $(u_n, v_n) \to (\phi, \psi)$ strongly in $\mathscr{D}$. Moreover, we have $\Phi_\varepsilon(\phi, \psi) = c_\varepsilon$ and $\Phi'_\varepsilon(\phi, \psi) = 0$. Thus, (ϕ, ψ) is a minimizer of c_ε. It is easy to see that $(|\phi|, |\psi|)$ is also a minimizer. Hence, without loss of generality, we may assume that $\phi \geq 0, \psi \geq 0$ and it follows that (ϕ, ψ) is a nonnegative solution of (8.7). Recalling (8.49), it is easy to see that $\phi \neq 0, \psi \neq 0$. Finally, by the strong maximum principle, we can obtain that $\phi > 0, \psi > 0$. That is, we obtain that (ϕ, ψ) is a positive ground state solution of (8.7). □

8.4 Geometric structure of positive ground state to (8.7)

Now, let us define the mountain pass value

$$\tilde{c}_\varepsilon := \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} \Phi_\varepsilon(\gamma(t)), \tag{8.50}$$

where $\Gamma_\varepsilon := \{\gamma(t) \in C([0,1], \mathscr{D}) : \gamma(0) = (0,0), \Phi_\varepsilon(\gamma(1)) < 0\}$. We have the following result.

Theorem 8.3. *Assume* $s_1, s_2 \in (0,2), \lambda, \mu \in (0, +\infty), \kappa > 0, \alpha > 1, \beta > 1$ *and* $\alpha + \beta = 2^*(s_2)$. *Let* $\varepsilon \in (0, s_2)$ *and one of the following hold:*

(i) $\lambda > \mu$ *and either* $1 < \beta < 2$ *or* $\begin{cases} \beta = 2 \\ \kappa > \eta_{1,\varepsilon} \end{cases}$ *;*

(ii) $\lambda = \mu$ *and either* $\min\{\alpha, \beta\} < 2$ *or* $\begin{cases} \min\{\alpha, \beta\} = 2, \\ \kappa > \eta_{1,\varepsilon} = \eta_{2,\varepsilon} \end{cases}$ *;*

(iii) $\lambda < \mu$ *and either* $1 < \alpha < 2$ *or* $\begin{cases} \alpha = 2 \\ \kappa > \eta_{2,\varepsilon} \end{cases}$ *.*

Then $c_\varepsilon = \tilde{c}_\varepsilon$ *and any positive ground state solution of system* (8.7) *is a mountain pass solution.*

Proof. It is easy to check that Φ_ε satisfies the mountain pass geometric structure. Recalling the existence result of Theorem 8.2, let (ϕ, ψ) be a positive ground state solution of (8.7). Define $\gamma_0(t) := tT(\phi, \psi)$ for some $T > 0$ large enough such that $\Phi_\varepsilon(T\phi, T\psi) < 0$. Then it is easy to see that $\gamma_0 \in \Gamma_\varepsilon$. By Lemma 8.1, we have

$$\Phi_\varepsilon(\phi, \psi) = \max_{t > 0} \Phi_\varepsilon(t\phi, t\psi). \tag{8.51}$$

Hence,

$$\tilde{c}_\varepsilon \leq \max_{t \in [0,1]} \Phi_\varepsilon(\gamma_0(t)) = \Phi_\varepsilon(\phi, \psi) = c_\varepsilon. \tag{8.52}$$

Under the assumptions, it is standard to prove that $\tilde{c}_\varepsilon$ is also a critical value and there exists a solution $(\tilde{\phi}, \tilde{\psi})$ such that $\Phi_\varepsilon(\tilde{\phi}, \tilde{\psi}) = \tilde{c}_\varepsilon$ and $\Phi'_\varepsilon(\tilde{\phi}, \tilde{\psi}) = 0$ in $\mathscr{D}^*$. Then we see that $(\tilde{\phi}, \tilde{\psi}) \in \mathcal{N}_\varepsilon$. Hence, by the definition of c_ε we see that

$$c_\varepsilon := \inf_{(u,v) \in \mathcal{N}_\varepsilon} \Phi_\varepsilon(u, v) \leq \Phi_\varepsilon(\tilde{\phi}, \tilde{\psi}) = \tilde{c}_\varepsilon. \tag{8.53}$$

By (8.52) and (8.53), we obtain that $c_\varepsilon = \tilde{c}_\varepsilon$. For any positive ground state solution, by the arguments as above, we have the mountain path $\gamma_0 \in \Gamma_\varepsilon$ and thus, the positive ground state is indeed a mountain pass solution. $\square$

Remark 8.4.

(i) *Recalling that for $\varepsilon \in [0, s_2)$, both $\eta_{1,\varepsilon}$ and $\eta_{2,\varepsilon}$ are increasing by ε and continuous with respect to ε. When $\kappa > \frac{\eta_{i,0}}{2^*(s_2)}, i \in \{1, 2\}$, then by the continuity, we see that $\kappa > \frac{\eta_{i,\varepsilon}}{2^*(s_2)}$ when ε is small enough.*

(ii) *Note that the proof of $\tilde{c}_\varepsilon = c_\varepsilon$ for $\varepsilon \in (0, s_2)$ depends heavily on the existence of the ground state solution. When $\varepsilon = 0$, the existence of ground state solution is still unknown. However, we will prove that the result $\tilde{c}_0 = c_0$ is also satisfied (see Corollary 8.2 below).*

Lemma 8.8. $\tilde{c}_\varepsilon \geq \tilde{c}_0$ *and* $\lim_{\varepsilon \to 0^+} \tilde{c}_\varepsilon = \tilde{c}_0$

Proof. By the monotonicity of $a_\varepsilon(x)$, it is easy to see that $\tilde{c}_\varepsilon \geq \tilde{c}_0$. Hence,

$$\lim_{\varepsilon \to 0^+} \tilde{c}_\varepsilon \geq \tilde{c}_0. \tag{8.54}$$

Next, we only need to prove the inverse inequality. For any $\delta > 0$, there exists $\gamma_0 \in \Gamma_0$ such that

$$\max_{t \in [0,1]} \Phi_0(\gamma_0(t)) < \tilde{c}_0 + \delta. \tag{8.55}$$

Denote $\gamma_0(1) := (\phi, \psi)$, since $\gamma_0 \in \Gamma_0$, we have $\Phi_0(\phi, \psi) < 0$.
Case 1: If $|\phi|^\alpha |\psi|^\beta \equiv 0$, it is easy to see that $\Phi_\varepsilon(\phi, \psi) = \Phi_0(\phi, \psi) < 0$. Hence, $\gamma_0 \in \Gamma_\varepsilon$ for all $\varepsilon \in [0, s_2)$ for this case.
Case 2: If $|\phi|^\alpha |\psi|^\beta \not\equiv 0$, then by the Lebesgue's dominated convergence theorem, we have

$$\lim_{\varepsilon \to 0^+} \int_\Omega a_\varepsilon(x) |\phi|^\alpha |\psi|^\beta dx = \int_\Omega a_0(x) |\phi|^\alpha |\psi|^\beta dx. \tag{8.56}$$

Hence, we have $\Phi_\varepsilon(\phi, \psi) < 0$ when ε is small enough. Thus, we also obtain that $\gamma_0 \in \Gamma_\varepsilon$ when ε is small enough. Now, we take an arbitrary sequence $\varepsilon_n \downarrow 0$ as $n \to +\infty$. Choose $t_n \in [0, 1]$ such that

$$\Phi_{\varepsilon_n}(\gamma_0(t_n)) = \max_{t \in [0,1]} \Phi_{\varepsilon_n}(\gamma_0(t)). \tag{8.57}$$

Up to a subsequence, we assume that $t_n \to t^* \in [0, 1]$ and denote that

$$\gamma_0(t_n) := (u_n, v_n), \gamma_0(t^*) := (u^*, v^*). \tag{8.58}$$

Since $\gamma_0 \in C([0, 1], \mathscr{D})$, we obtain that $(u_n, v_n) \to (u^*, v^*)$ and it follows that

$$\Phi_{\varepsilon_n}(u_n, v_n) = \Phi_{\varepsilon_n}(u^*, v^*) + o(1). \tag{8.59}$$

By the Lebesgue's dominated convergence theorem again, we have

$$\Phi_{\varepsilon_n}(u^*, v^*) = \Phi_0(u^*, v^*) + o(1). \tag{8.60}$$

Hence, by (8.59) and (8.60), we have $\Phi_{\varepsilon_n}(u_n,v_n)=\Phi_0(u^*,v^*)+o(1)$. Then

$$\begin{aligned}\tilde{c}_{\varepsilon_n} \leq &\Phi_{\varepsilon_n}(\gamma_0(t_n))=\Phi_{\varepsilon_n}(u_n,v_n)\\ =&\Phi_0(u^*,v^*)+o(1)=\Phi_0(\gamma_0(t^*))+o(1)\\ \leq &\max_{t\in[0,1]}\Phi_0(\gamma_0(t))+o(1)=\Phi_0(\phi,\psi)+o(1)\\ \leq &\tilde{c}_0+\delta+o(1).\end{aligned} \tag{8.61}$$

Let $n\to+\infty$, we obtain that $\lim\limits_{n\to+\infty}\tilde{c}_{\varepsilon_n}\leq\tilde{c}_0+\delta$. Hence, $\lim_{\varepsilon\to0^+}\tilde{c}_\varepsilon\leq\tilde{c}_0$. Insert (8.54), we complete the proof. □

Corollary 8.2. $c_0=\tilde{c}_0$ *and* $\lim\limits_{\varepsilon\to0^+}c_\varepsilon=c_0$.

Proof. For any $(u,v)\neq(0,0)$, define $\gamma(t)=t(u,v)$, then we see that $\gamma\in\Gamma_0$. Hence, it is easy to see that $\tilde{c}_0\leq c_0$. On the other hand, by Theorem 8.3 and Lemma 8.8, we have $\tilde{c}_0=\lim_{\varepsilon\to0^+}\tilde{c}_\varepsilon=\lim_{\varepsilon\to0^+}c_\varepsilon$. By Lemma 8.3, we have $\lim_{\varepsilon\to0^+}c_\varepsilon\geq c_0$. Hence, we obtain that $\tilde{c}_0=c_0$ and $\lim\limits_{\varepsilon\to0^+}c_\varepsilon=c_0$. □

8.5 The existence of the positive ground state to the original system

Take $\{\varepsilon_n\}\subset(0,s_2)$ such that $\varepsilon_n\downarrow0$ as $n\to+\infty$. By Theorem 8.2, the system (8.7) possesses a positive ground state solution (u_n,v_n). By Remark 8.2, we have

$$c_{\varepsilon_n}=\Phi_{\varepsilon_n}(u,v)\geq\Big(\frac{1}{2}-\frac{1}{2^*(s_{max})}\Big)\big(\|u_n\|^2+\|v_n\|^2\big). \tag{8.62}$$

By Corollary 8.2, we have $c_{\varepsilon_n}\to c_0$. Hence, $\{(u_n,v_n)\}$ is bounded in $\mathscr{D}$. Up to a subsequence, we may assume that $(u_n,v_n)\rightharpoonup(u_0,v_0)$ weakly in $\mathscr{D}$ and $u_n\to u_0, v_n\to v_0$ a.e. in Ω. We shall establish the following results which are useful to prove our main theorem.

Lemma 8.9. (u_0,v_0) *satisfies* $\Phi_0'(u_0,v_0)=0$ *in* $\mathscr{D}^*$.

Proof. We claim that for any $\phi\in D_0^{1,2}(\Omega)$, we have

$$\lim_{n\to+\infty}\int_\Omega a_{\varepsilon_n}(x)|u_n|^{\alpha-2}u_n|v_n|^\beta\phi dx=\int_\Omega a_0(x)|u_0|^{\alpha-2}u_0|v_0|^\beta\phi dx. \tag{8.63}$$

Without loss of generality, we may also assume that $\phi\geq0$. If not, we view $\phi=\phi_+-\phi_-$, and we discuss ϕ_+ and ϕ_- respectively.

Firstly, by Fatou's Lemma, we have

$$\int_\Omega a_0(x)|u_0|^{\alpha-2}u_0|v_0|^\beta\phi dx\leq\liminf_{n\to+\infty}\int_\Omega a_{\varepsilon_n}(x)|u_n|^{\alpha-2}u_n|v_n|^\beta\phi dx. \tag{8.64}$$

On the other hand, since $a_{\varepsilon_n}(x)\leq a_0(x)$, we have

$$\int_\Omega a_{\varepsilon_n}(x)|u_n|^{\alpha-2}u_n|v_n|^\beta\phi dx\leq\int_\Omega a_0(x)|u_n|^{\alpha-2}u_n|v_n|^\beta\phi dx. \tag{8.65}$$

Further, since $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in $\mathscr{D}$, it is easy to see that

$$|u_n|^{\alpha-2}u_n|v_n|^{\beta} \rightharpoonup |u_0|^{\alpha-2}u_0|v_0|^{\beta} \quad \text{in} \quad L^{\frac{2^*(s_2)}{2^*(s_2)-1}}(\Omega, a_0(x)dx),$$

then we have

$$\lim_{n\to+\infty}\int_\Omega a_0(x)|u_n|^{\alpha-2}u_n|v_n|^{\beta}\phi dx = \int_\Omega a_0(x)|u_0|^{\alpha-2}u_0|v_0|^{\beta}\phi dx. \tag{8.66}$$

By (8.65) and (8.66), we have

$$\limsup_{n\to+\infty}\int_\Omega a_{\varepsilon_n}(x)|u_n|^{\alpha-2}u_n|v_n|^{\beta}\phi dx \le \int_\Omega a_0(x)|u_0|^{\alpha-2}u_0|v_0|^{\beta}\phi dx. \tag{8.67}$$

Hence, from (8.64) and (8.67), we prove (8.63).

Similarly, we can prove that for any $\psi \in D_0^{1,2}(\Omega)$, we have

$$\lim_{n\to+\infty}\int_\Omega a_{\varepsilon_n}(x)|u_n|^{\alpha}|v_n|^{\beta-2}v_n\psi dx = \int_\Omega a_0(x)|u_0|^{\alpha}|v_0|^{\beta-2}v_0\psi dx. \tag{8.68}$$

Recalling that (u_n, v_n) are critical point of Φ_{ε_n}, for any $(\phi, \psi) \in \mathscr{D}$, we have

$$\left\langle \Phi'_{\varepsilon_n}(u_n, v_n), (\phi, \psi)\right\rangle \equiv 0. \tag{8.69}$$

Then by (8.63), (8.68) and $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in $\mathscr{D}$, we obtain that

$$\left\langle \Phi'_0(u_0, v_0), (\phi, \psi)\right\rangle = 0. \tag{8.70}$$

Hence, $\Phi'_0(u_0, v_0) = 0$ in $\mathscr{D}^*$. $\square$

Lemma 8.10. *If $(u_0, v_0) \neq (0, 0)$, then $\Phi_0(u_0, v_0) = c_0 > 0$.*

Proof. Since (u_n, v_n) is a positive ground state solution of (P_{ε_n}), it is easy to prove that

$$c_{\varepsilon_n} = \Phi_{\varepsilon_n}(u_n, v_n) = \Big[\frac{1}{2} - \frac{1}{2^*(s_1)}\Big]b(u_n, v_n) + \Big[\frac{2^*(s_2)}{2} - 1\Big]\kappa c_{\varepsilon_n}(u_n, v_n). \tag{8.71}$$

By Lemma 8.9, we also have

$$\Phi_0(u_0, v_0) = \Big[\frac{1}{2} - \frac{1}{2^*(s_1)}\Big]b(u_0, v_0) + \Big[\frac{2^*(s_2)}{2} - 1\Big]\kappa c_0(u_0, v_0). \tag{8.72}$$

Noting that by Fatou's Lemma, we have

$$b(u_0, v_0) \le \liminf_{n\to+\infty} b(u_n, v_n) \tag{8.73}$$

and

$$c_0(u_0, v_0) \le \liminf_{n\to+\infty} c_{\varepsilon_n}(u_n, v_n). \tag{8.74}$$

Hence, $\Phi_0(u_0, v_0) \leq \lim_{n\to+\infty} c_{\varepsilon_n}$ and $\Phi_0(u_0, v_0) \geq 0$ follows by (8.72). By Corollary 8.2, we have $\lim_{n\to+\infty} c_{\varepsilon_n} = c_0$, hence

$$\Phi_0(u_0, v_0) \leq c_0. \tag{8.75}$$

On the other hand, when $(u_0, v_0) \neq (0, 0)$, it is trivial that $\Phi_0(u_0, v_0) \geq c_0 > 0$ by Lemma 8.9 and the definition of c_0. Hence, we obtain that $\Phi_0(u_0, v_0) = c_0$ if $(u_0, v_0) \neq (0, 0)$. □

Lemma 8.11.

$$\begin{aligned}\Phi_0(u_0, v_0) &< \min\{m_\lambda, m_\mu\}\\ &= \left[\frac{1}{2} - \frac{1}{2^*(s_1)}\right] \Big(\mu_{s_1}(\Omega)\Big)^{\frac{2^*(s_1)}{2^*(s_1)-2}} \Big(\max\{\lambda, \mu\}\Big)^{-\frac{2}{2^*(s_1)-2}}.\end{aligned}$$

Proof. It is a direct conclusion by Lemma 8.7 and Lemma 8.10. □

Corollary 8.3. *If* $(u_0, v_0) \neq (0, 0)$*, then* (u_0, v_0) *is a positive ground solution of* (1.1)*.*

Proof. Since (u_n, v_n) are positive and $u_n \to u_0, v_n \to v_0$ a.e. in Ω. We see that $u_0 \geq 0, v_0 \geq 0$. If $v_0 = 0$, then we see that $\Psi'_\lambda(u_0) = 0$ and $u_0 \neq 0$. Hence, $\Phi_0(u_0, v_0) = \Psi_\lambda(u_0) \geq m_\lambda$, a contradiction to Lemma 8.11. Similarly, if $u_0 = 0, v_0 \neq 0$, we see that $\Phi_0(u_0, v_0) \geq m_\mu$, also a contradiction to Lemma 8.11. Hence, $u_0 \neq 0, v_0 \neq 0$ and $\Phi_0(u_0, v_0) = c_0$ by Lemma 8.10. That is, (u_0, v_0) is a nontrivial and nonnegative ground state solution of (1.1). Finally, by the strong maximum principle, we can prove that (u_0, v_0) is a positive solution. □

Lemma 8.12. *Assume that* $\liminf_{n\to+\infty} \int_\Omega a_{\varepsilon_n} |u_n|^\alpha |v_n|^\beta = 0$*, then* $\Psi'_\lambda(u_n) \to 0, \Psi'_\mu(v_n) \to 0$ *in* $H^{-1}(\Omega)$*.*

Proof. Under the assumptions, we claim that up to a subsequence,

$$\int_\Omega \left|a_{\varepsilon_n}(x)|u_n|^{\alpha-2}u_n|v_n|^\beta h\right| dx = o(1)\|h\|. \tag{8.76}$$

For any $h \in D_0^{1,2}(\Omega)$, since $a_{\varepsilon_n}(x) \leq a_0(x)$, we have

$$\int_\Omega a_{\varepsilon_n} |h|^{2^*(s_2)} dx \leq \int_\Omega a_0 |h|^{2^*(s_2)} dx.$$

Then by the Hardy-Sobolev inequality, we obtain that there exists some $C > 0$ independent of n such that

$$\Big(\int_\Omega a_{\varepsilon_n} |h|^{2^*(s_2)} dx\Big)^{\frac{1}{2^*(s_2)}} \leq C\|h\|. \tag{8.77}$$

Noting that $\frac{\alpha-1}{\alpha}+\frac{\beta}{2^*(s_2)\alpha}+\frac{1}{2^*(s_2)}=1$, by Hölder inequality and (8.77), we have

$$\begin{aligned}
&\int_\Omega \Big|a_{\varepsilon_n}(x)|u_n|^{\alpha-2}u_n|v_n|^\beta h\Big|dx\\
&\le \Big(\int_\Omega a_{\varepsilon_n}|u_n|^\alpha|v_n|^\beta dx\Big)^{\frac{\alpha-1}{\alpha}}\\
&\quad \Big(\int_\Omega a_{\varepsilon_n}(x)|v_n|^{2^*(s_2)}dx\Big)^{\frac{\beta}{2^*(s_2)\alpha}}\ \Big(\int_\Omega a_{\varepsilon_n}|h|^{2^*(s_2)}dx\Big)^{\frac{1}{2^*(s_2)}}\\
&= o(1)\|h\|,
\end{aligned}\tag{8.78}$$

which means that (8.76) is proved. Since (u_n, v_n) is a positive ground state solution of the system (8.7), we have

$$\big\langle \Phi'_{\varepsilon_n}(u_n,v_n),(h,0)\big\rangle \equiv 0. \tag{8.79}$$

That is,

$$\int_\Omega \Big[\nabla u_n\nabla h-\lambda\frac{|u_n|^{2^*(s_1)-1}u_n}{|x|^{s_1}}h-\kappa\alpha\int_\Omega a_{\varepsilon_n}(x)|u_n|^{\alpha-2}u_n|v_n|^\beta h\Big]dx\equiv 0 \tag{8.80}$$

for all n and $h\in D_0^{1,2}(\Omega)$. Then by (8.76) and (8.80), we obtain that

$$\int_\Omega \Big[\nabla u_n\nabla h-\lambda\frac{|u_n|^{2^*(s_1)-1}u_n}{|x|^{s_1}}h\Big]dx=o(1)\|h\|. \tag{8.81}$$

Hence, $\Psi'_\lambda(u_n)\to 0$ in $H^{-1}(\Omega)$. Similarly, we can prove that $\Psi'_\mu(v_n)\to 0$ in $H^{-1}(\Omega)$. $\square$

Corollary 8.4. $\liminf\limits_{n\to+\infty}\int_\Omega a_{\varepsilon_n}|u_n|^\alpha|v_n|^\beta>0$.

Proof. By Lemma 8.2, we see that

$$\|u_n\|^2+\|v_n\|^2\ge\delta^2_{\varepsilon_n}\ge\delta_0^2>0. \tag{8.82}$$

Hence, we obtain that

$$(u_n,v_n)\not\to(0,0)\text{ in }\mathscr{D}. \tag{8.83}$$

If $\liminf\limits_{n\to+\infty}\int_\Omega a_{\varepsilon_n}|u_n|^\alpha|v_n|^\beta=0$, by Lemma 8.12, we obtain that $\Psi'_\lambda(u_n)\to 0$, $\Psi'_\mu(v_n)\to 0$ in $H^{-1}(\Omega)$. Since either $u_n\not\to 0$ or $v_n\not\to 0$, it is easy to see that either

$$\lim_{n\to+\infty}\Psi_\lambda(u_n)\ge m_\lambda$$

or

$$\lim_{n\to+\infty}\Psi_\mu(v_n)\ge m_\mu.$$

By the assumption of $\liminf\limits_{n\to+\infty}\int_\Omega a_{\varepsilon_n}|u_n|^\alpha|v_n|^\beta=0$ again, we have

$$\lim_{n\to+\infty}\Phi_{\varepsilon_n}(u_n,v_n)=\lim_{n\to+\infty}\Psi_\lambda(u_n)+\Psi_\mu(v_n)\ge\min\{m_\lambda,m_\mu\}, \tag{8.84}$$

a contradiction to Lemma 8.11. Thereby this corollary is proved. $\square$

Lemma 8.13. *Assume that* $\{(\phi_n, \psi_n)\}$ *is a bounded sequence of* $\mathscr{D}$ *such that*

$$\lim_{n\to+\infty} J_{\varepsilon_n}(\phi_n, \psi_n) = 0 \tag{8.85}$$

and

$$\liminf_{n\to+\infty} c_{\varepsilon_n}(\phi_n, \psi_n) > 0, \tag{8.86}$$

where the functionals $J_\varepsilon(u, v)$ *and* $c_\varepsilon(u, v)$ *are defined in* (8.11) *and* (8.10)*, respectively. Then*

$$\liminf_{n\to+\infty} \Phi_{\varepsilon_n}(\phi_n, \psi_n) \geq c_0. \tag{8.87}$$

Proof. Since $\liminf\limits_{n\to+\infty} c_{\varepsilon_n}(\phi_n, \psi_n) > 0$, we see that $(\phi_n, \psi_n) \not\to (0, 0)$ in $\mathscr{D}$. Without loss of generality, we may assume that $(\phi_n, \psi_n) \neq (0, 0)$ for all n. Combining with the boundedness of $\{(\phi_n, \psi_n)\}$, we obtain that there exists some $d_0, d_1 > 0$ such that

$$0 < d_0 \leq a(\phi_n, \psi_n) := \|\phi_n\|^2 + \|\psi_n\|^2 \leq d_1 \text{ for all } n. \tag{8.88}$$

We also claim that $b(\phi_n, \psi_n)$ is bounded and away from 0, i.e., there exists $d_3, d_4 > 0$ such that

$$0 < d_3 \leq b(\phi_n, \psi_n) := \lambda|\phi_n|^{2^*(s_1)}_{2^*(s_1),s_1} + \mu|\psi_n|^{2^*(s_1)}_{2^*(s_1),s_1} \leq d_4. \tag{8.89}$$

The right-hand inequality in (8.89) is trivial due to the Hardy-Sobolev inequality. Now, we only need to prove the existence of d_3. We proceed by contradiction. If $b(\phi_n, \psi_n) \to 0$ up to a subsequence, then $\phi_n \to 0, \psi_n \to 0$ strongly in $L^{2^*(s_1)}(\Omega, \frac{dx}{|x|^{s_1}})$. Recalling the boundedness of $\{(\phi_n, \psi_n)\}$ again, by Proposition 2.1 and Proposition 2.2, we obtain that $\phi_n \to 0, \psi_n \to 0$ strongly in $L^{2^*(s_2)}(\Omega, a_0(x)dx)$. Noting that $a_{\varepsilon_n}(x) \leq a_0(x)$, then by the Hölder inequality, it is easy to prove that

$$c_{\varepsilon_n}(\phi_n, \psi_n) \leq c_0(\phi_n, \psi_n) \to 0, \tag{8.90}$$

a contradiction to (8.86) and thereby (8.89) is proved. We also note that $c_{\varepsilon_n}(\phi_n, \psi_n)$ is bounded. Hence, up to a subsequence, we may assume that

$$a(\phi_n, \psi_n) \to a^* > 0, b(\phi_n, \psi_n) \to b^* > 0, c_{\varepsilon_n}(\phi_n, \psi_n) \to c^* > 0. \tag{8.91}$$

Then by (8.85), we see that

$$a^* - b^* - 2^*(s_2)\kappa c^* = 0. \tag{8.92}$$

On the other hand, for any n, by Lemma 8.1, there exists a unique $t_n > 0$ such that $J_{\varepsilon_n}(t_n\phi_n, t_n\psi_n) = 0$ and t_n is implicity given by the following equation

$$a(\phi_n, \psi_n) - b(\phi_n, \psi_n)t_n^{2^*(s_1)-2} - 2^*(s_2)\kappa c_{\varepsilon_n}(\phi_n, \psi_n)t_n^{2^*(s_2)-2} = 0. \tag{8.93}$$

Since $a(\phi_n, \psi_n)$ is bounded and $\liminf\limits_{n\to+\infty} c_{\varepsilon_n}(\phi_n, \psi_n) > 0$, it is easy to see that t_n is bounded. On the other hand, by the Hardy-Sobolev inequality again, we obtain that

$$a(\phi_n, \psi_n) \leq C_1\big(a(\phi_n, \psi_n)\big)^{\frac{2^*(s_1)}{2}} t_n^{2^*(s_1)-2} + C_2\big(a(\phi_n, \psi_n)\big)^{\frac{2^*(s_2)}{2}} t_n^{2^*(s_2)-2}, \tag{8.94}$$

for some positive constants C_1, C_2 independent of n. Then it is easy to see that at least one of the following holds:

(i) $\frac{1}{2}a(\phi_n,\psi_n) \le C_1\big(a(\phi_n,\psi_n)\big)^{\frac{2^*(s_1)}{2}} t_n^{2^*(s_1)-2}$;

(ii) $\frac{1}{2}a(\phi_n,\psi_n) \le C_2\big(a(\phi_n,\psi_n)\big)^{\frac{2^*(s_2)}{2}} t_n^{2^*(s_2)-2}$.

Hence, we see that t_n is bounded away from 0. Up to a subsequence if necessary, we assume that $t_n \to t^* > 0$. Then we have that

$$\left.\begin{array}{l} J_{\varepsilon_n}(t_n\phi_n, t_n\psi_n) \equiv 0, \\ \{(\phi_n,\psi_n)\} \text{ is bounded in } \mathscr{D}, \\ t_n \to t^* > 0, \end{array}\right\} \Rightarrow \lim_{n\to+\infty} J_{\varepsilon_n}(t^*\phi_n, t^*\psi_n) = 0.$$

By (8.91), we obtain that

$$a^*(t^*)^2 - b^*(t^*)^{2^*(s_1)} - 2^*(s_2)\kappa c^*(t^*)^{2^*(s_2)} = 0, \quad (t^* > 0). \tag{8.95}$$

It is easy to see that the algebraic equation $a^* = b^* t^{2^*(s_1)-2} + c^* t^{2^*(s_2)-2}$ has a unique positive solution. Hence, by (8.92) and (8.95), we obtain that $t^* = 1$. Then recalling the boundedness of $\{(\phi_n,\psi_n)\}$ again, we see that

$$\lim_{n\to+\infty} \Phi_{\varepsilon_n}(\phi_n,\psi_n) = \lim_{n\to+\infty} \Phi_{\varepsilon_n}(t_n\phi_n, t_n\psi_n). \tag{8.96}$$

By the definition of t_n, we see that $(t_n\phi_n, t_n\psi_n) \in \mathcal{N}_{\varepsilon_n}$. Hence, $\Phi_{\varepsilon_n}(t_n\phi_n, t_n\psi_n) \ge c_{\varepsilon_n}$. Then by (8.96) and Corollary 8.2, we obtain that

$$\lim_{n\to+\infty} \Phi_{\varepsilon_n}(\phi_n,\psi_n) \ge \lim_{n\to+\infty} c_{\varepsilon_n} = c_0. \tag{8.97}$$

□

Lemma 8.14. *Assume* $s_1, s_2 \in (0,2), \lambda, \mu \in (0,+\infty), \kappa > 0, \alpha > 1, \beta > 1$ *and* $\alpha + \beta = 2^*(s_2)$. *Let* $\varepsilon \in (0, s_2)$, *then any solution* (u,v) *of the system* (8.7) *satisfies*

$$\int_{\Omega\cap\mathbb{B}_1} \kappa a_\varepsilon(x)|u|^\alpha|v|^\beta dx = \int_{\Omega\cap\mathbb{B}_1^c} \kappa a_\varepsilon(x)|u|^\alpha|v|^\beta dx, \tag{8.98}$$

where $\mathbb{B}_1$ *is the unit ball of* $\mathbb{R}^N$ *entered at zero.*

Proof. Let

$$G(x,u,v) = \frac{\lambda}{2^*(s_1)}\frac{|u|^{2^*(s_1)}}{|x|^{s_1}} + \frac{\mu}{2^*(s_1)}\frac{|v|^{2^*(s_1)}}{|x|^{s_1}} + \kappa a_\varepsilon(x)|u|^\alpha|v|^\beta. \tag{8.99}$$

Noting that

$$\frac{\partial}{\partial x_i} a_\varepsilon(x) = \begin{cases} -(s_2-\varepsilon)\frac{1}{|x|^{s_2+2-\varepsilon}} x_i & \text{for } |x| < 1, \\ -(s_2+\varepsilon)\frac{1}{|x|^{s_2+2+\varepsilon}} x_i & \text{for } |x| > 1, \end{cases} \tag{8.100}$$

we have

$$
\begin{aligned}
& x_i \cdot G_{x_i}(x,u,v) \\
=& \begin{cases} -s_1\Big[\frac{\lambda}{2^*(s_1)}\frac{|u|^{2^*(s_1)}}{|x|^{s_1}}+\frac{\mu}{2^*(s_1)}\frac{|v|^{2^*(s_1)}}{|x|^{s_1}}\Big]\frac{x_i^2}{|x|^2}-(s_2-\varepsilon)\frac{\kappa|u|^\alpha|v|^\beta}{|x|^{s_2+2-\varepsilon}}x_i^2 & \text{if } |x|<1, \\ -s_1\Big[\frac{\lambda}{2^*(s_1)}\frac{|u|^{2^*(s_1)}}{|x|^{s_1}}+\frac{\mu}{2^*(s_1)}\frac{|v|^{2^*(s_1)}}{|x|^{s_1}}\Big]\frac{x_i^2}{|x|^2}-(s_2+\varepsilon)\frac{\kappa|u|^\alpha|v|^\beta}{|x|^{s_2+2+\varepsilon}}x_i^2 & \text{if } |x|>1. \end{cases}
\end{aligned}
\tag{8.101}
$$

Hence, by (7.29), we have

$$
\begin{aligned}
& -2(N-s_1)\int_\Omega\big[\frac{\lambda}{2^*(s_1)}\frac{|u|^{2^*(s_1)}}{|x|^{s_1}}+\frac{\mu}{2^*(s_1)}\frac{|v|^{2^*(s_1)}}{|x|^{s_1}}\big]dx \\
& -2(N-s_2)\int_\Omega \kappa a_\varepsilon(x)|u|^\alpha|v|^\beta dx \\
& +(N-2)\int_\Omega(|\nabla u|^2+|\nabla v|^2)dx \\
=& 2\varepsilon\int_{\Omega\cap\mathbb{B}_1}\kappa a_\varepsilon(x)|u|^\alpha|v|^\beta dx-2\varepsilon\int_{\Omega\cap\mathbb{B}_1^c}\kappa a_\varepsilon(x)|u|^\alpha|v|^\beta dx.
\end{aligned}
\tag{8.102}
$$

On the other hand, since (u,v) is a solution of the system (8.7), we have

$$
\begin{aligned}
& \int_\Omega\big(|\nabla u|^2+|\nabla v|^2\big)dx \\
=& \int_\Omega\big(\lambda\frac{|u|^{2^*(s_1)}}{|x|^{s_1}}+\mu\frac{|v|^{2^*(s_1)}}{|x|^{s_1}}+2^*(s_2)\kappa a_\varepsilon(x)|u|^\alpha|v|^\beta\big)dx.
\end{aligned}
\tag{8.103}
$$

Since $\varepsilon>0$, by (8.102) and (8.103), we obtain the result of (8.98). □

Proof of Theorem 8.1: By Corollary 8.3, we only need to prove that $(u_0,v_0)\neq(0,0)$. Now,we proceed by contradiction. We assume that $(u_0,v_0)=(0,0)$. By Corollary 8.4, we have, up to a subsequence if necessary, that

$$\lim_{n\to+\infty}\int_\Omega a_{\varepsilon_n}(x)|u_n|^\alpha|v_n|^\beta dx := \tau>0. \tag{8.104}$$

On the other hand, by Corollary 8.14, we have

$$\int_{\Omega\cap\mathbb{B}_1} a_{\varepsilon_n}(x)|u_n|^\alpha|v_n|^\beta dx=\int_{\Omega\cap\mathbb{B}_1^c} a_{\varepsilon_n}(x)|u_n|^\alpha|v_n|^\beta dx \quad \text{for all } n. \tag{8.105}$$

Hence,

$$\lim_{n\to+\infty}\int_{\Omega\cap\mathbb{B}_1} a_{\varepsilon_n}(x)|u_n|^\alpha|v_n|^\beta dx=\lim_{n\to+\infty}\int_{\Omega\cap\mathbb{B}_1^c} a_{\varepsilon_n}(x)|u_n|^\alpha|v_n|^\beta dx=\frac{\tau}{2}>0. \tag{8.106}$$

Let $\chi(x) \in C_c^\infty(\mathbb{R}^N)$ be a cut-off function such that $\chi(x) \equiv 1$ in $\mathbb{B}_{\frac{1}{2}}$, $\chi(x) \equiv 0$ in $\mathbb{R}^N \backslash \mathbb{B}_1$ and take $\tilde{\chi}(x) \in C^\infty(\mathbb{R}^N)$ such that $\tilde{\chi}(x) \equiv 0$ in $\mathbb{B}_1$ and $\tilde{\chi} \equiv 1$ in $\mathbb{R}^N \backslash \mathbb{B}_2$. Denote

$$\begin{cases} \phi_{1,n}(x) := \chi(x)u_n(x), \ \phi_{2,n}(x) := \tilde{\chi}(x)u_n(x), \\ \psi_{1,n}(x) := \chi(x)v_n(x), \ \psi_{2,n}(x) := \tilde{\chi}(x)v_n(x). \end{cases} \tag{8.107}$$

Recalling that (u_n, v_n) is a positive ground state solution of the system (8.7) with $\varepsilon = \varepsilon_n$, then

$$\left\langle \Phi'_{\varepsilon_n}(u_n, v_n), (u_n - \phi_{1,n} - \phi_{2,n}, v_n - \psi_{1,n} - \psi_{2,n}) \right\rangle \equiv 0 \text{ for all } n. \tag{8.108}$$

when $(u_0, v_0) = (0, 0)$, by Rellich-Kondrachov compactness theorem, if $0 \notin \overline{\tilde{\Omega}}$, we have that $u_n \to 0, v_n \to 0$ strongly in $L^{2^*(s_1)}(\tilde{\Omega}, \frac{dx}{|x|^{s_1}})$ and $L^{2^*(s_2)}(\tilde{\Omega}, a_{\varepsilon_n}dx)$ uniformly for all n. Hence, it is easy to prove that

$$(u_n - \phi_{1,n} - \phi_{2,n}, v_n - \psi_{1,n} - \psi_{2,n}) \to (0, 0) \text{ strongly in } \mathscr{D}. \tag{8.109}$$

We also have that

$$\left\langle \Phi'_{\varepsilon_n}(u_n, v_n), (\phi_{1,n}, \psi_{1,n}) \right\rangle \equiv 0 \text{ for all } n. \tag{8.110}$$

Then by $(u_0, v_0) = (0, 0)$ and Rellich-Kondrachov compactness theorem again, it is easy to see that

$$\lim_{n\to+\infty} J_{\varepsilon_n}(\phi_{1,n}, \psi_{1,n}) = 0. \tag{8.111}$$

We also obtain that

$$\liminf_{n\to+\infty} c_{\varepsilon_n}(\phi_{1,n}, \psi_{1,n}) = \lim_{n\to+\infty} \int_{\Omega\cap\mathbb{B}_1} a_{\varepsilon_n}(x)|u_n|^\alpha |v_n|^\beta dx = \frac{\tau}{2} > 0. \tag{8.112}$$

Hence, by Lemma 8.13, we have

$$\lim_{n\to+\infty} \Phi_{\varepsilon_n}(\phi_{1,n}, \psi_{1,n}) \geq c_0. \tag{8.113}$$

Similarly, we can prove that

$$\lim_{n\to+\infty} \Phi_{\varepsilon_n}(\phi_{2,n}, \psi_{2,n}) \geq c_0. \tag{8.114}$$

By $(u_0, v_0) = (0, 0)$ and the Rellich-Kondrachov compact theorem again, we have that

$$c_{\varepsilon_n} = \lim_{n\to+\infty} \Phi_{\varepsilon_n}(u_n, v_n) = \lim_{n\to+\infty} \left[\Phi_{\varepsilon_n}(\phi_{1,n}, \psi_{1,n}) + \Phi_{\varepsilon_n}(\phi_{2,n}, \psi_{2,n}) \right]. \tag{8.115}$$

Then by (8.113),(8.114) and Corollary 8.2, we obtain that

$$c_0 \geq 2c_0, \tag{8.116}$$

a contradict to the fact of that $c_0 > 0$. Hence, $(u_0, v_0) \neq (0, 0)$ and the proof is completed.

References

[1] B. Abdellaoui, V. Felli and I. Peral, Some remarks on systems of elliptic equations doubly critical in the whole $\mathbb{R}^N$, *Calculus of variations and partial differential equations*, **34**(1): 97-137, 2009.

[2] N. Akhmediev and A. Ankiewicz, Partially coherent solitons on a finite background. *Physical review letters*, **82**(13): 2661, 1999.

[3] A. Ambrosetti and E. Colorado, Bound and ground states of coupled nonlinear schrödinger equations, *Comptes Rendus Mathematique*, **342**(7): 453-458, 2006.

[4] H. Brezis and T. Kato, Remarks on the schroedinger operator with singular complex potentials. Technical report, DTIC Document, 1978.

[5] L. Caffarelli, R. Kohn and L. Nirenberg, First order interpolation inequalities with weights, *Compositio Mathematica*, **53**(3): 259-275, 1984.

[6] P. Caldiroli and R. Musina, On the existence of extremal functions for a weighted Sobolev embedding with critical exponent, *Calc. Var. Partial Differential Equations*, **8**(4): 365-387, 1999.

[7] F. Catrina and Z.-Q. Wang, On the caffarelli-kohn-nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions, *Communications on Pure and Applied Mathematics*, **54**(2): 229-258, 2001.

[8] G. Cerami, X. Zhong and W. Zou, On some nonlinear elliptic pdes with sobolev-hardy critical exponents and a Li-Lin open problem, *Calculus of Variations and Partial Differential Equations*, to appear, 2015.

[9] J.-L. Chern and C.-S. Lin, Minimizers of caffarelli–kohn–nirenberg inequalities with the singularity on the boundary, *Archive for Rational Mechanics and Analysis*, **197**(2): 401-432, 2010.

[10] H. Egnell, Positive solutions of semilinear equations in cones. *Transactions of the American Mathematical Society*, **330**(1): 191-201, 1992.

[11] S. G. Evangelides Jr, L. F. Mollenauer, J. P. Gordon and N. S. Bergano, Polarization multiplexing with solitons, *Lightwave Technology, Journal of*, **10**(1): 28-35, 1992.

[12] N. Ghoussoub and X. Kang, Hardy–sobolev critical elliptic equations with boundary singularities, In *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, volume 21, pages 767-793. Elsevier, 2004.

[13] N. Ghoussoub and F. Robert, The effect of curvature on the best constant in the hardy–sobolev inequalities, *Geometric & Functional Analysis GAFA*, **16**(6): 1201-1245, 2006.

[14] N. Ghoussoub and C. Yuan, Multiple solutions for quasi-linear pdes involving the critical sobolev and hardy exponents, *Transactions of the American Mathematical Society*, **352**(12): 5703-5743, 2000.

[15] C.-H. Hsia, C.-S. Lin, and H. Wadade, Revisiting an idea of brézis and nirenberg, *Journal of Functional Analysis*, **259**(7): 1816-1849, 2010.

[16] I. P. Kaminow, Polarization in optical fibers, *IEEE Journal of Quantum Electronics*, **17**: 15-22, 1981.

[17] Y. Li and C.-S. Lin, A nonlinear elliptic pde with two sobolev–hardy critical exponents, *Archive for Rational Mechanics and Analysis*, **203**(3): 943-968, 2012.

[18] E. H. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, *Ann. of Math. (2)*, **118**(2): 349-374, 1983.

[19] C.-S. Lin, H. Wadade, et al, Minimizing problems for the hardy-sobolev type inequality with the singularity on the boundary, *Tohoku Mathematical Journal*, **64**(1): 79-103, 2012.

[20] T.-C. Lin and J. Wei, Ground state of n coupled nonlinear schrödinger equations in $\mathbb{R}^n, n \leq 3$, *Communications in mathematical physics*, **255**(3): 629-653, 2005.

[21] P.-L. Lions, The concentration-compactness principle in the calculus of variations, the limit case, part 2. *Rev. Mat. Iberoamericana*, **1**(2): 45-121, 1985.

[22] L. Maia, E. Montefusco, and B. Pellacci, Positive solutions for a weakly coupled nonlinear schrödinger system, *Journal of Differential Equations*, **229**(2): 743-767, 2006.

[23] C. R. Menyuk, Nonlinear pulse propagation in birefringent optical fibers, *Quantum Electronics, IEEE Journal of*, **23**(2): 174-176, 1987.

[24] C. R. Menyuk, Pulse propagation in an elliptically birefringent kerr medium, *Quantum Electronics, IEEE Journal of*, **25**(12): 2674-2682, 1989.

[25] B. Sirakov, Least energy solitary waves for a system of nonlinear schrödinger equations in $\mathbb{R}^N$, *Communications in mathematical physics*, **271**(1): 199-221, 2007.

[26] M. Struwe, *Variational methods: applications to nonlinear partial differential equations and Hamiltonian systems*, volume 34. Springer, 2008.

[27] G. Talenti, Best constant in Sobolev inequality, *Ann. Mat. Pura Appl. (4)*, **110**: 353-372, 1976.

[28] P. Wai, C. R. Menyuk and H. Chen, Stability of solitons in randomly varying birefringent fibers, *Optics letters*, **16**(16): 1231-1233, 1991.